NTNU

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Department of Mathematical Sciences

Examination paper for
TMA4220 Numerical Solution of Partial Differential Equations Using Element Methods

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Examination date: 13th of December 2021

Examination time (from–to): 09:00–13:00

Permitted examination support material: C:

- A. Quarteroni: *Numerical Models for Differential Problems, Springer.*
- S. Brenner and L. R. Scott: *The Mathematical Theory of Finite Element Methods, Springer.*
- *TMA4220 Lecture Notes Fall 2021* (Front page + 229 pages)
- *TMA4220 2021H AFEM* (Front page + 43 pages)
- Rottmann: *Matematisk formelsamling*
- Approved calculator

Other information:
All answers should be justified and include enough details to make it clear which methods and/or results have been used. All the (sub-)problems are worth 5 points each. The total value is 65 points.

Language: English

Number of pages: 3

Number of pages enclosed: 0

Checked by:

Informasjon om trykking av eksamensoppgave
Originalen er:
1-sidig ☐ **2-sidig** ☒
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Date Signature

Problem 1 Consider the two-dimensional steady heat equation

$$
\begin{aligned}
-\nabla(\kappa\nabla u) &= f \quad \text{in} \quad \Omega & (1)\\
u &= 0 \quad \text{on} \quad \partial\Omega_D & (2)\\
H(u) &= \bar{t} \quad \text{on} \quad \partial\Omega_N. & (3)
\end{aligned}
$$

where κ is the conductivity, u is the unknown temperature, f the applied loading, $\bar{u}$ and $\bar{t}$ are respectively prescribed Dirichlet and Neumann boundary conditions for the given steady heat problem defined on the polygonal domain $\Omega \in R^2$. The differential operator $H(u)$ is the so called "Neumann operator".

Assume that $\kappa = \kappa(x, y) > \kappa_{\min} > 0$ for all $(x, y) \in \Omega$.

a) Use Galerkin's method and establish the weak formulation corresponding to the problem (1)-(3) on the form: Find $u \in X$ such that

$$
a(u, v) = l(v) \qquad \forall v \in X \tag{4}
$$

In particular, identify X, a and l for this problem. Identify also the expression for the "Neumann operator" $H(u)$ related to the Neumann boundary conditions.

b) What conditions have to be fulfilled such that there exists a unique solution of the weak form in (4).

c) Assume that we choose the finite element method to solve (4) numerically. Formulate the corresponding finite element variational formulation.

d) Explain how you will handle inhomogeneous Dirichlet boundary conditions, i.e., $u = \bar{u}$ *on* $\partial\Omega_\mathrm{D}$ instead of Equation (2) in a finite element program.

e) Given the unit square $\Omega = (0, 1) * (0, 1)$, assume homogeneous Dirichlet condition on the boundary $\partial\Omega$, constant loading f, and a constant conductivity κ. Establish the linear algebraic system of equations in the form: $\mathbf{Ax} = \mathbf{b}$, where $\mathbf{A}$ is the system coefficient matrix, $\mathbf{x}$ is the vector of unknown nodal coefficients and $\mathbf{b}$ is the right hand for the two different meshes with linear triangular elements shown in Figure (1). Compute the resulting finite element temperature field and visualize the temperature variation along the two diagonals.

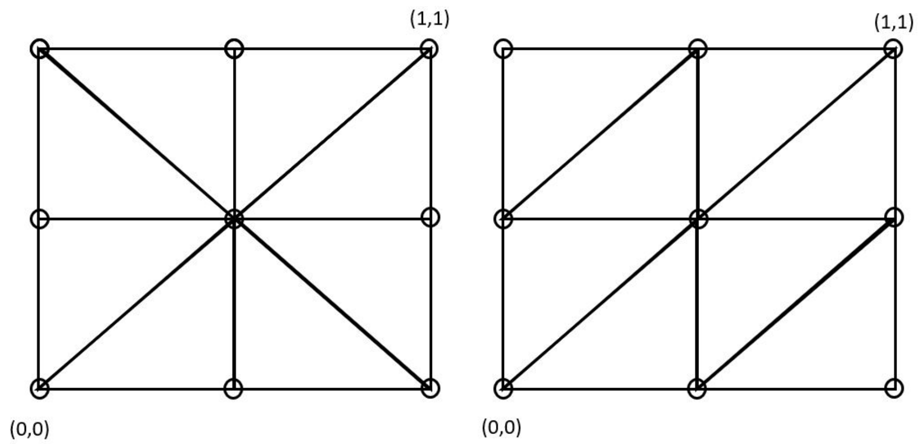

Figure 1: Left: Mesh 1. Right: Mesh 2.

Problem 2

a) What characterize a finite element? What is a compatible finite element?

b) Define a compatible quadrilateral Lagrange type finite element with 5 nodes as displayed in Figure 2, where the interior node is in the center.

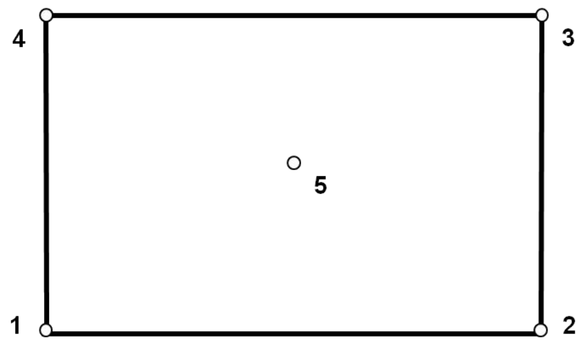

Figure 2: Compatible quadratilateral element with 5 nodes.

c) Define a compatible cubic (i.e. 3^{rd} order) quadrilateral Lagrange type finite element, i.e., show all the nodal locations and the corresponding element nodal basis functions for one vertex, one edge node and one interior node (i.e., three in total).

d) Define Galerkin orthogonality. Which property of the finite element method is induced by Galerkin orthogonality?

e) Define the nodal basis functions for the 4 node "transfer element" shown in Figure 3, such that it is compatible with 6 node quadratic Lagrange triangular element along the edge with the midpoint node, and compatible with neighbouring 3 node linear Lagrange triangular elements along the two other edges.

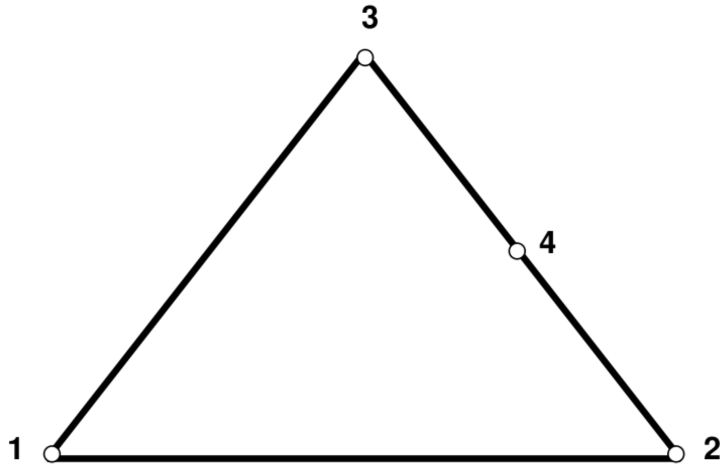

Figure 3: "Transfer element" with 4 nodes.

Problem 3 Here we again consider a two dimensional Poisson problem as described in Problem 1, i.e., Equation (1)–(3) with the weak formulation as given in Equation (4) to be solved with a corresponding compatible finite element problem using the finite dimensional space X_h.

a) Will the resulting finite element solution be optimal in any norm? If yes, explain which norm it is and what is the underlying criteria for it to happen.

b) Set up an *a priori error estimate* for the error in the natural norm for the given Poisson problem.

c) Explain the main steps involved in adaptive finite element methods (AFEM). Why do we want to use AFEM?