Norwegian University of Science
and Technology
Department of Mathematical
Sciences

TMA4220 Numerical Solution of Partial Differential Equations Using Element Methods Autumn 2020

Exercise set 2 Solution

1 **a)** Since we are changing the physical interval $[a, b]$ into the reference interval $[0, 1]$, the transformation formulas for u and its inverse become

$$u(x) = \frac{1}{h}(x - a) \quad , \quad x(u) = hu + a$$

where $h = b - a$. Inserting this into the exact integral yields

$$\begin{aligned}\int_a^b f(x)\,dx &= \int_0^1 f(hu + a)h\,du \\ &\approx h\sum_{i=1}^{m} w_i f(hu_i + a)\end{aligned}$$

where u_i and w_i are respectively the *shifted* quadrature nodes and weights (**Note:** Gaussian quadrature is defined on $[-1, 1]$. Transition to $[0, 1]$ requires a slight modification $w_i = \widehat{w}_i/2$ and $x_i = (\widehat{x}_i + 1)/2$).

b) From the chain rule, we obtain

$$\frac{df}{dx} = \frac{df}{du}\frac{du}{dx} = \frac{1}{h}\frac{df}{du}$$

Inserting this into the exact integral yields

$$\begin{aligned}\int_a^b f'(x)g'(x)\,dx &= \int_0^1 \frac{1}{h}f'(hu + a)\frac{1}{h}g'(hu + a)h\,du \\ &\approx \frac{1}{h}\sum_{i=1}^{m} w_i f'(hu_i + a)g'(hu_i + a)\end{aligned}$$

c) The formulas for the indices in the matrices are as follows:

$$\widehat{\mathbf{M}}_e^{(i,j)} = \int_{-1}^{1} \varphi_i\varphi_j\,dx \quad , \quad \widehat{\mathbf{K}}_e^{(i,j)} = \int_{-1}^{1} \frac{d\varphi_i}{dx}\frac{d\varphi_j}{dx}\,dx$$

By direct calculation of the integrals, we get

$$\widehat{\mathbf{M}}_e = \frac{1}{6}\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix} \quad , \quad \widehat{\mathbf{K}}_e = \begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$$

d) By direct calculation of the integrals as above, we get

$$\widehat{\mathbf{M}}_e = \frac{1}{30}\begin{bmatrix}4 & 2 & -1\\ 2 & 16 & 2\\ -1 & 2 & 4\end{bmatrix} \quad , \quad \widehat{\mathbf{K}}_e = \frac{1}{3}\begin{bmatrix}7 & -8 & 1\\ -8 & 16 & -8\\ 1 & -8 & 7\end{bmatrix}$$

2 Our uniform partition implies that we can express the grid as

$$a = x_0 < x_1 < x_2 < \cdots < x_{N-1} < x_N = b$$

where $x_i - x_{i-1} = h$ and $h = (b-a)/N$. So if $I_i = [x_{i-1}, x_i]$, then the local mass and stiffness matrices on this element is given by

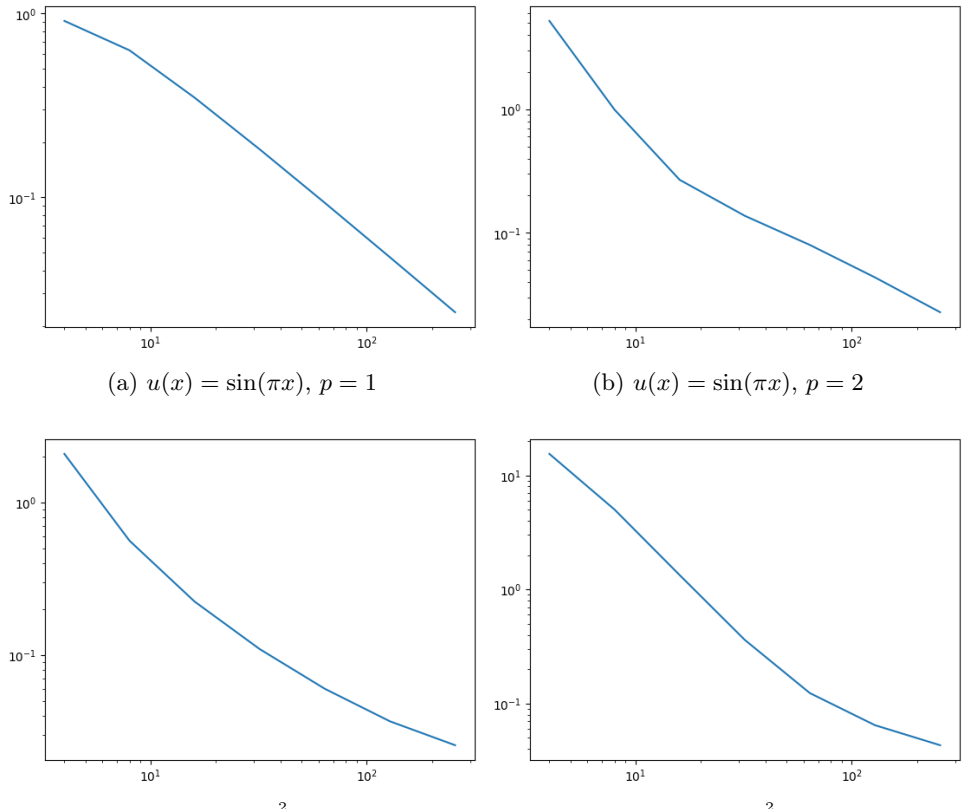
$$\mathbf{M}_e = h\widehat{\mathbf{M}}_e \quad , \quad \mathbf{K}_e = \frac{1}{h}\widehat{\mathbf{K}}_e$$

The elements are C^0-continuous, so the overlapping is at the first and last entries of each local matrix, and the dimension of the global system matrix is $(Np+1)\times(Np+1)$. The local element vectors are defined as

$$\mathbf{f}_e = h\sum_{i=1}^{m} w_i f(hu_i + x_i)\boldsymbol{\varphi}(hu_i + x_i)$$

where $\boldsymbol{\varphi} = [\varphi_0(x), \varphi_1(x), \ldots, \varphi_p(x)]^T$. When we calculate the global error, we pick out each element and the corresponding local coefficient vectors. Hence, the square of the local element error becomes

$$e_i^2 = h\sum_{i=1}^{m} w_i \left(\mathbf{u}_{I_i} \cdot \boldsymbol{\varphi}(hu_i + x_i) - u(hu_i + x_i)\right)^2$$



(a) $u(x) = \sin(\pi x)$, $p = 1$ (b) $u(x) = \sin(\pi x)$, $p = 2$

(c) $u(x) = e^{-x^2}$, $p = 1$ (d) $u(x) = e^{-x^2}$, $p = 2$

Figure 1: Convergence plot of the simulations.