Norwegian University of Science and Technology
Department of Mathematical Sciences

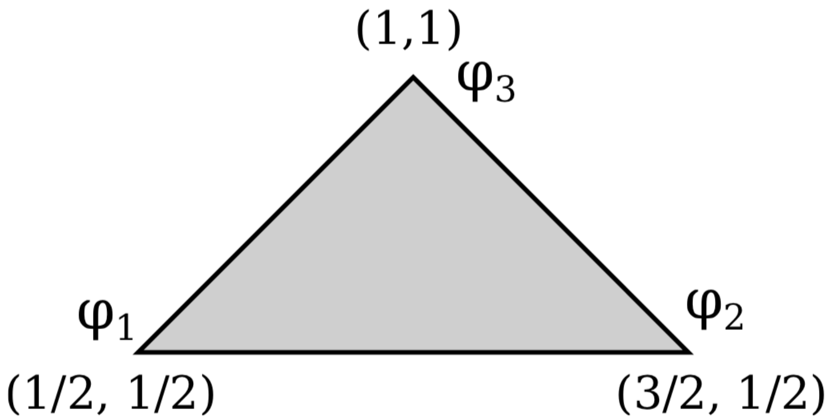
TMA4220 Numerical Solution of Partial Differential Equations Using Element Methods Fall 2020

Exercise set 3

1 Consider the triangle with corners $(\frac{1}{2}, \frac{1}{2})$, $(1, 1)$ and $(\frac{3}{2}, \frac{1}{2})$. The linear functions on this triangle can be written as

$$\phi_i(x, y) = a_i x + b_i y + c.$$

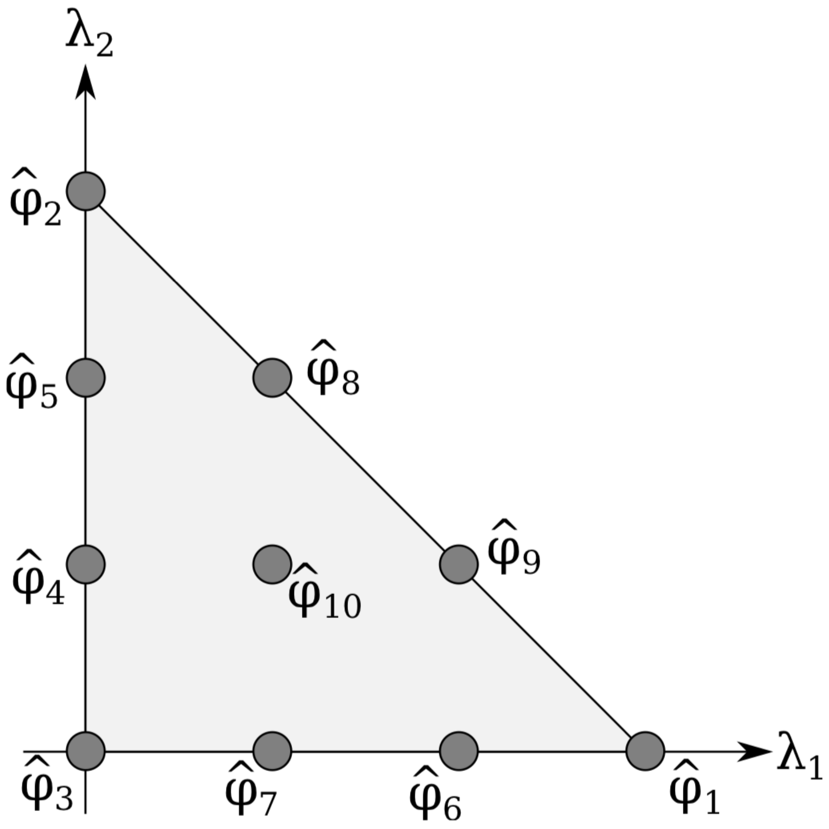
Find the expression for the three basis functions on this element in physical coordinates (x, y).



2 Consider the 10-node reference triangle of unit length: X_h^3. The *cubic* functions can be written as

$$\hat{\phi}_i(\lambda_1, \lambda_2, \lambda_3) = \sum_{0 \le i+j+k \le 3} a_{ijk} \lambda_1^i \lambda_2^j \lambda_3^k.$$

Find the expression fot the ten basis functions on this element in barycentric (area) coordinates $(\lambda_1, \lambda_2, \lambda_3)$.



3 **The Poincaré inequality:** In this exercise you will prove Poincaré's inequality: If $\Omega \subset \mathbb{R}^d$ is an open, bounded domain, then there exists a constant $C > 0$ such that

$$\|u\|_{L^2(\Omega)} \leq C|u|_{H^1(\Omega)} \tag{1}$$

for every $u \in H_0^1(\Omega)$. Here $|u|_{H^1(\Omega)}$ denotes the H^1 seminorm

$$|u|_{H^1(\Omega)} := \|\nabla u\|_{L^2(\Omega)} = \Big(\int_\Omega \sum_{i=1}^d \Big|\frac{\partial u}{\partial x_i}\Big|^2 \, dx\Big)^{1/2}.$$

Informally, this means that in H_0^1 on bounded domains, you can bound the size of a function by its derivative.

a) We assume first that $\Omega = [-M, M]^2$, the box in $\mathbb{R}^2$ with center 0 and side lengths $2M$, for some positive M. Show that there is a $C > 0$ such that for all $u \in H_0^1(\Omega)$,

$$|u(x)|^2 \leq C \int_{-M}^{M} \Big|\frac{\partial u}{\partial x_1}(y_1, x_2)\Big|^2 d\,y_1, \qquad \text{for all } x = (x_1, x_2) \in \Omega.$$

Hint: Use the fundamental theorem of calculus and Cauchy's inequality.

b) Integrate over x_2 and show that

$$\int_{-M}^{M} |u(x_1, x_2)|^2 d\,x_2 \leq C \int_\Omega \Big|\frac{\partial u}{\partial x_1}(y)\Big|^2 d\,y \qquad \text{for all } x_1 \in [-M, M].$$

c) Now integrate over x_1 and conclude with (1).

d) If $\Omega \subset \mathbb{R}^2$ is a general open, bounded domain, you can find some $M > 0$ such that $\Omega \subset [-M, M]^2$ (why?). For a function $u \in H_0^1(\Omega)$, find some function $\bar{u} \in H_0^1([-M, M])$ such that $u = \bar{u}$ in Ω and

$$\|\bar{u}\|_{L^2([-M,M]^2)} = \|u\|_{L^2(\Omega)}$$

and

$$|\bar{u}|_{H^1([-M,M]^2)} = |u|_{H^1(\Omega)}.$$

Explain why this implies that the Poincaré inequality also holds on Ω.

4 (Exam 2019) Consider the one-dimensional Poisson equation

$$-u_{xx} = f \quad \text{in} \quad \Omega \tag{2}$$

with the boundary conditions

$$u = 0 \quad \text{on} \quad \partial\Omega_D \tag{3}$$
$$H(u) = \bar{t} \quad \text{on} \quad \partial\Omega_N. \tag{4}$$

where u is the unknown solution, f the applied loading and $\bar{t}$ are prescribed Neumann boundary conditions for the given Poisson problem defined on the interval $\Omega \in R^1$. The differential operator $H(u)$ is the so called "Neumann operator".

a) Use Galerkin's method and establish the weak formulation corresponding to the problem (2)-(4) on the form: Find $u \in X$ such that

$$a(u, v) = l(v) \qquad \forall v \in X \tag{5}$$

In particular, identify X, a and l for this problem. Identify also the expression for the "Neumann operator" $H(u)$ related to the Neumann boundary conditions.

b) Assume that we choose the finite element method to solve (5) numerically. Formulate the corresponding finite element variational formulation.

c) Establish the linear algebraic system of equations in the form: $\mathbf{Ax} = \mathbf{b}$, where $\mathbf{A}$ is the system coefficient matrix (also denoted "stiffness matrix"), $\mathbf{x}$ is the vector of unknown nodal coefficients and $\mathbf{b}$ is the right hand side due to applied loading and prescribed boundary conditions given in (2)-(4).

d) Introduce quadratic Lagrange nodal basis functions to express the finite element solution u_h and the corresponding test function v_h for the finite element variational problem above. Compute the coefficient matrix ("element stiffness matrix") $\mathbf{A_e}$ for one quadratic Lagrange 1D-element (line element with three nodes).

e) Compute the system coefficient matrix $\mathbf{A}$ for the above finite element Poisson problem where $\Omega = (0, 1)$, $\partial\Omega_D = \{x = 0\}$ and $\partial\Omega_N = \{x = 1\}$ that is discretized with three quadratic Lagrange 1D-elements of equal length.

5 (Exam 2019) Here we again consider a one-dimensional Poisson problem as described in Problem 1, i.e., Equation (1)–(3) with the weak formulation as given in Equation (4) to be solved with a corresponding compatible finite element problem using the finite dimensional space X_h.

a) What is the superconvergence property? What is the conditions for it to be present?

Hint: read the AFEM notes posted at Blackboard.

b) State the three conditions for the recovery operator G_{x_h} that is sufficient to guarantee that $G_{\mathrm{x}_h}(I_{\mathrm{x}_h} u)$ is a good approximation to the true gradient ∇u. Here I_{x_h} is the interpolant operator on the finite element space X_h.

c) Construct a recovery operator G_{x_h} that fulfills the three conditions found in 3(b) for one-dimensional linear finite elements. Check that your recovery operator is *consistent*.

d) Describe how you can utilise the recovered gradient $G_{\mathrm{x}_h}(u_h)$ to estimate the error in the numerically computed finite element solution $u_h \in X_h$.