Norwegian University of Science and Technology
Department of Mathematical Sciences

TMA4220
Numerical Solution of Partial Differential Equations Using Element Methods
Fall 2020

Solutions to exercise set 3

1 We known that $\phi_i(\mathbf{x_j}) = \delta_{ij}$, where $\mathbf{x_j}, j = 1, 2, 3$ are the three corners, and we know that the linear basis functions are on the form

$$\phi_i(x, y) = a_i x + b_i y + c_i.$$

For ϕ_1 we get the system

$$\begin{aligned} \phi_i(1/2, 1/2) = 1/2a_1 + 1/2b_1 + c_1 = 1, \\ \phi_i(3/2, 1/2) = 3/2a_1 + 1/2b_1 + c_1 = 0, \\ \phi_i(1, 1) = a_1 + b_1 + c_1 = 0. \end{aligned}$$

If we set up the same systems for ϕ_2 and ϕ_3 we get

$$\begin{bmatrix} 1/2 & 1/2 & 1 \\ 3/2 & 1/2 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

with solution

$$\begin{aligned} \phi_1(x, y) &= -x - y + 2, \\ \phi_2(x, y) &= x - y, \\ \phi_3(x, y) &= 2y - 1. \end{aligned}$$

2 We first start with the basis functions that are 1 in the corners, i.e. ϕ_1, ϕ_2 and ϕ_3. For these basis functions, we can write them on the form $\phi_i = d_i(\lambda_i - a_i)(\lambda_i - b_i)(\lambda_i - c_i)$ (why?). As we want the functions to be zero in all other nodes, we must have, for $i = 1$, $a_1 = 0$, $b_1 = 1/3$, $c_1 = 2/3$. Then, since $\phi_1(1, 0, 0) = 1$, $d_1 = (1 - 0)(1 - 1/3)(1 - 2/3) = 2/9$. Thus

$$\hat{\phi}_1 = 9/2\lambda_1(\lambda_1 - 1/3)(\lambda_1 - 2/3).$$

By symmetry, we find

$$\hat{\phi}_2 = 9/2\lambda_2(\lambda_2 - 1/3)(\lambda_2 - 2/3),$$

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For $\hat{\phi}_4$ we note that this has to be zero along the lines through $\hat{\phi}_1$ and $\hat{\phi}_2$, and through $\hat{\phi}_5$ and $\hat{\phi}_6$. Also, $\hat{\phi}_4(0, 2/3, 1/3) = 2/27$, so $d_4 = 27/2$. Thus, we have

$$\hat{\phi}_4 = 27/2\lambda_2\lambda_3(\lambda_3 - 1/3).$$

By symmetry,

$$\begin{aligned}
\hat{\phi}_5 &= 27/2\lambda_2\lambda_3(\lambda_2 - 1/3),\\
\hat{\phi}_6 &= 27/2\lambda_3\lambda_1(\lambda_1 - 1/3),\\
\hat{\phi}_7 &= 27/2\lambda_3\lambda_1(\lambda_3 - 1/3),\\
\hat{\phi}_8 &= 27/2\lambda_1\lambda_2(\lambda_2 - 1/3),\\
\hat{\phi}_9 &= 27/2\lambda_1\lambda_2(\lambda_1 - 1/3).
\end{aligned}$$

We see that $\hat{\phi}_{10}$ must be zero along all the edges, which yields

$$\hat{\phi}_{10} = 27\lambda_1\lambda_2\lambda_3.$$

3 **a)** By the fundamental theorem of calculus,

$$|u(x)|^2 = \Big|\int_{-M}^{x} \frac{\partial u}{\partial x_1}(y_1, x_2)\,d\,y_1\Big|^2,$$

and using Cauchy's inequality with $f = \frac{\partial u}{\partial x_1}(y_1, x_2)$ and $g = 1$, we get

$$\begin{aligned}
\Big|\int_{-M}^{x} \frac{\partial u}{\partial x_1}(y_1, x_2)\,d\,y_1\Big|^2 &\le \int_{-M}^{x} \Big|\frac{\partial u}{\partial x_1}(y_1, x_2)\Big|^2 d\,y_1 \cdot \int_{-M}^{x} dx\\
&\le 2M \int_{-M}^{M} \Big|\frac{\partial u}{\partial x_1}(y_1, x_2)\Big|^2 d\,y_1.
\end{aligned}$$

b) Integrate over x_2 to get

$$\int_{-M}^{M} |u(x)|^2 d\,x_2 \le 2M \int_{-M}^{M}\int_{-M}^{M} \Big|\frac{\partial u}{\partial x_1}(y_1, x_2)\Big|^2 d\,y_1 d\,x_2 = 2M \int_{\Omega} \Big|\frac{\partial u}{\partial x_1}(y)\Big|^2 d\,y,$$

where the equality comes from Fubini-Tonell's theorem.

c) Integrate over x_1 so

$$\begin{aligned}
\|u\|^2_{L^2(\Omega)} = \int_{\Omega} |u(x)|^2\,dx = \int_{-M}^{M}\int_{-M}^{M} |u(x)|^2 d\,x_2 d\,x_1\\
&\le 2M \int_{-M}^{M}\int_{\Omega} \Big|\frac{\partial u}{\partial x_1}(y)\Big|^2 d\,y d\,x_1\\
&= 4M^2 \int_{\Omega} \Big|\frac{\partial u}{\partial x_1}(y)\Big|^2 d\,y\\
&\le 4M^2 \int_{\Omega} \Big|\frac{\partial u}{\partial x_1}(y)\Big|^2 + \Big|\frac{\partial u}{\partial x_2}(y)\Big|^2 d\,y\\
&= 4M^2 |u|^2_{H^1(\Omega)}.
\end{aligned}$$

d) Note that since Ω is bounded its diameter is finite. Let $D = \text{diam}(\Omega) < \infty$. Choose a point $x \in \Omega$ and let $L = \|x\|$. Then $\Omega \subset [-(D+L),(D+L)]^2$. The function $\bar{u}$ defined as

$$\bar{u}(x) = \begin{cases} u(x) & \text{if } x \in \Omega \\ 0 & \text{else} \end{cases}$$

satisfies the requirements.

Remarks:

- The Poincaré inequality on bounded domains $\Omega \subset \mathbb{R}^d$ is shown in an analogous fashion.
- Note that the constant C grows as the box becomes larger.
- If $u \in H_0^k(\Omega)$ for any $k \geq 1$, then $\frac{\partial u}{\partial x_i} \in H_0^{k-1}(\Omega)$ for any $i = 1, \dots, d$, and more generally, $D^\alpha u \in H_0^{k-|\alpha|}$ for any multiindex α of size $\alpha \leq k$. Thus, we can iterate Poincaré inequality and find that there is a constant $C > 0$ such that

$$\|u\|_{L^2(\Omega)} \leq C|u|_{H^1(\Omega)} \leq \cdots \leq C|u|_{H^k(\Omega)}, \forall u \in H_0^k(\Omega).$$

4 See solutions to exam 2019.