Norwegian University of Science and Technology
Department of Mathematical Sciences

TMA4220
Numerical Solution of Partial Differential Equations Using Element Methods
Fall 2020

Solutions to exercise set 1

1 **a)** (i) Compute the $L^2([-1,1])$ norm

$$\int_{-1}^{1}(|x|^\alpha)^2\,dx = 2\int_0^1 |x|^{2\alpha}\,dx = \frac{2}{2\alpha+1}[x^{2\alpha+1}]_0^1.$$

This blows up at 0 if $2\alpha+1<0$. If $2\alpha+1=0$ we get $\log x$, which also blows up at 0. Thus, $|x|^\alpha \in L^2([-1,1])$ if $\alpha > -\frac{1}{2}$.

(ii) Similarly we have

$$\int_1^\infty |x|^{2\alpha}\,dx = \lim_{y\to\infty}\frac{1}{2\alpha+1}[x^{2\alpha+1}]_1^y,$$

so we need $\alpha < -1/2$.

(iii) Transform to polar coordinates

$$\int_{B_1(0)} |x|^{2\alpha}\,dx = \int_0^{2\pi}\int_0^1 r^{2\alpha+1}d\,r\,d\,\theta,$$

and we see that the inner integral is the same integral as in (i). Thus we need $\alpha > -1$.

b) Since D is closed and bounded and f is continuous, f takes its maximum value M on D. Then

$$\int_D f^2\,dx \le \int_D M^2\,dx = M^2\mu(D),$$

where $\mu(D)$ is the size of D.

c) (i) Assume both v_1 and v_2 are weak derivatives of the same function u. Then, by the definition of the weak derivative

$$\int_\Omega (v_1 - v_2)\phi\,dx = \int_\Omega (u-u)\phi'\,dx = 0.$$

Hence $v_1 = v_2$ almost everywhere, i.e. equal in L^2.

(ii) If $u \in C^1$, we have

$$-\int u'\phi\,dx = \int u\phi'\,dx$$

by integration by parts. The boundary terms vanish since $\phi \in C_c^\infty(\Omega)$.

d) (i) Compute the two integrals as

$$\int_0^2 f_1^2\,dx = \int_0^1 x^2\,dx + \int_1^2 dx = 4/3 < \infty$$
$$\int_0^2 f_2^2\,dx = \int_0^1 x^2\,dx + \int_1^2 2\,dx = 7/3 < \infty,$$

so $f_1, f_2 \in L^2$.

(ii) Compute the weak derivative of f_1 by computing

$$\begin{aligned}\int_0^2 f_1\phi'\,dx &= \int_0^1 x\phi'(x)\,dx + \int_1^2 \phi'(x)\,dx \\ &= [x\phi(x)]_0^1 - \int_0^1 \phi\,dx + [\phi(x)]_1^2 \\ &= \phi(1) + \phi(2) - \phi(1) - \int_0^1 \phi(x)\,dx - \int_1^2 0\phi(x)\,dx.\end{aligned}$$

As $\phi(2) = 0$ since ϕ has compact support in $[0, 2]$, the weak derivative must be

$$v_1(x) = \begin{cases} 1 & \text{if } 0 < x < 1, \\ 0 & \text{if } 1 \leq x < 2, \end{cases}$$

(ii) For f_2 we see that it is not continuous, so the weak derivative can not exist (since by the Soboloev embedding theorem $H^1 \subset C^0$). Another way to see this is to compute $\int_0^2 u\phi'\,dx$ and note that this will always depend on $\phi(1)$.

(iii) To show that $f_1 \notin H^2(\Omega)$, we note that the weak derivative v_1 of f_1 is not continuous.

2 **a)** Using the same discrete test space X_h^1 as for the Dirichlet problem, the weak formulation becomes

$$\begin{aligned}\int_0^1 u'(x)v'(x)\,dx - [u'(x)v(x)]_0^1 &= \int_0^1 fv\,dx \\ \int_0^1 u'(x)v'(x)\,dx - u'(1)v(1) + u'(0)v(0) &= \int_0^1 fv\,dx \\ \int_0^1 u'(x)v'(x)\,dx &= \int_0^1 fv\,dx + 1\end{aligned}$$

by inserting $v(0) = 0$, $v(1) = 1$ and $u'(1) = 1$. Then the new linear system becomes

$$Au = f + [0, \ldots, 1]^T.$$

The code can then be modified by removing the boundary condition for the right end point, and adding 1 to the last element in the b-vector.

b) Integrate twice to obtain

$$\begin{aligned}u'(x) &= -x + A \\ u(x) &= -\frac{1}{2}x^2 + Ax + B.\end{aligned}$$

Use the boundary conditions

$$
\begin{aligned}
u(0) = 0 &\implies B = 0 \\
u'(1) = 1 &\implies A = 1 + 1 = 2 \\
u(x) &= -\frac{x^2}{2} + 2x
\end{aligned}
$$

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a) False. If $v_1, v_2 \in \mathcal{S}$, then $v_1 + v_2 \notin \mathcal{S}$ since $(v_1 + v_2)(\frac{1}{2}) = 1 + 1 = 2 \neq 1$.

b) True. L maps elements in X to elements in $\mathbb{R}$, and $L(\alpha_1 v_1 + \alpha_2 v_2) = \alpha_1 L(v_1) + \alpha_2 L(v_2)$ for all $\alpha_1, \alpha_2 \in \mathbb{R}$ and all $v_1, v_2 \in X$.

c) False. $(x, y)_X$ is not bilinear since $(x_1 + x_2, y)_X = |x_1 + x_2||y| \neq |x_1||y| + |x_2||y|$ if x_1 and x_2 has opposite signs.

d) False. Use $w(x) = 1$.

4

a) Choose a test function $v \in V$, integrate by parts twice:

$$
\begin{aligned}
\int_0^1 u_{xxxx} v \, dx &= [u_{xxx} v]_0^1 - \int_0^1 u_{xxx} v_x \, dx \\
&= [u_{xxx} v]_0^1 - [u_{xx} v_x]_0^1 + \int_0^1 u_{xx} v_{xx} \, dx \\
&= \int_0^1 f v \, dx.
\end{aligned}
$$

Since $v(0) = v(1) = v_x(0) = v_x(1) = 0$, the two boundary terms vanish, and we end up with

$$
a(u, v) = \int_0^1 u_{xx} v_{xx} \, dx
$$

and

$$
f(v) = \int_0^1 f v \, dx.
$$

Then the bilinear form a is symmetric and positive since $a(v, v) = \int_0^1 v_{xx}^2 \, dx \geq 0$ with $a(v, v) = 0$ only if $v_{xx} = 0$, which implies that $v = 0$ because of the boundary conditions.

b) $V = H_0^2((0, 1)) = \{v \in H^2((0, 1)) \mid v(0) = v(1) = v_x(0) = v_x(1) = 0\}$.