TMA4220
Numerical Solution of Partial Differential Equations Using Element Methods
Autumn 20200

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Exercise set 2

1 **a)** Consider the finite integral, where $h = b - a$:

$$\int_a^b f(x)\,dx$$

You are going to integrate it with Gaussian quadrature. How will the general formula for the approximated integral be if you have m quadrature points, and choose the reference interval as $[0,1]$?

b) Consider the new finite integral:

$$\int_a^b f'(x)g'(x)\,dx$$

How will you modify the previous formula in order to use Gaussian quadrature?

c) Consider the linear basis functions on the reference element $\widehat{\Omega} = [0,1]$:

$$\varphi_0(x) = 1 - x \quad , \quad \varphi_1(x) = x$$

Calculate the element mass and stiffness matrices on this interval, $\widehat{\mathbf{M}}_e$ and $\widehat{\mathbf{K}}_e$.

d) Consider the quadratic basis functions on the reference element $\widehat{\Omega} = [0,1]$:

$$\varphi_0(x) = (x-1)(2x-1) \quad , \quad \varphi_1(x) = 4(1-x)x \quad , \quad \varphi_2(x) = x(2x-1)$$

Calculate the element mass and stiffness matrices on this interval, $\widehat{\mathbf{M}}_e$ and $\widehat{\mathbf{K}}_e$.

2 Consider the BVP

$$-u_{xx} + \sigma u = f \quad , \quad u(a) = u_1 \ , \ u(b) = u_2 \tag{1}$$

Construct a finite element code which can solve this BVP with linear or quadratic elements. This program requires the following input parameters:

- p (polynomial degree of basis functions).
- N (number of elements on the uniform interval).
- σ, a, b, u_1, u_2 (BVP parameters).

Use Gaussian quadrature with 5 points for assembling the load vector $\mathbf{f}$. Recall that the elements are C^0-continuous, and this will influence the overlapping between elements in the global mass and stiffness matrices $\mathbf{M}$ and $\mathbf{K}$.

For the rest of the task, let $N = \{4, 8, 16, 32, 64, 128, 256\}$ and $p = \{1, 2\}$. Calculate the error in the continuous L^2-norm (store them in .txt-files) and create convergence plots (the other code opens the .txt-file and creates a loglog-plot). Recall that

$$E = \sqrt{\int_a^b |u(x_i) - u_h(x_i)|^2}$$

This quantity can also be calculated with Gaussian quadrature.

a) For the first simulation, let $\sigma = 0$, $a = 0$, $b = 2$, $u_1 = 0$, $u_2 = 0$, $u(x) = \sin(\pi x)$ and $f(x) = \pi^2 \sin(\pi x)$.

b) For the second simulation, let $\sigma = 2$, $a = -2$, $b = 2$, $u_1 = e^{-4}$, $u_2 = e^{-4}$, $u(x) = e^{-x^2}$ and $f(x) = 4(1 - x^2)e^{-x^2}$.