Norwegian University of Science and Technology
Department of Mathematical Sciences

TMA4220 Numerical Solution of Partial Differential Equations Using Element Methods Fall 2020

Exercise set 1

1 **a)** For which $\alpha \in \mathbb{R}$ does the function $f(x) := |x|^\alpha$ lie in respectively $L^2([-1,1])$, $L^2([1,\infty))$ and $L^2(B_1(0))$, where $B_1(0) = \{x \in \mathbb{R}^2 : |x| < 1\}$ is the unit ball?

b) Let $D \subset \mathbb{R}$ be a closed, bounded subset of $\mathbb{R}$ and $f \in C^0(D)$, show that $f \in L^2(D)$.

c) Let $\Omega \subset \mathbb{R}$ be some open interval. A *weak derivative* of a function $u : \Omega \to \mathbb{R}$ is a function $v : \Omega \to \mathbb{R}$ such that

$$\int_\Omega u(x)\phi'(x)\,dx = -\int_\Omega v(x)\phi(x)\,dx$$

for every $\phi \in C_c^\infty(\Omega)$, the set of infinitely differentiable functions with compact support in Ω. Show that the weak derivative (if it exists) is unique. Show that if u is continuously differentiable (i.e. $u \in C^1(\Omega)$), then $\frac{du}{dx}$ is its weak derivative.

d) Let

$$f_1(x) = \begin{cases} x & \text{if } 0 < x < 1, \\ 1 & \text{if } 1 \le x < 2, \end{cases}$$

$$f_2(x) = \begin{cases} x & \text{if } 0 < x < 1, \\ 2 & \text{if } 1 \le x < 2, \end{cases}$$

for $x \in \Omega := (0,2)$. Show that $f_1, f_2 \in L^2(\Omega)$. Show that $f_1 \in H^1(\Omega)$ by finding its weak derivative, and that $f_1 \notin H^2(\Omega)$. Show that $f_2 \notin H1(\Omega)$.

2 We consider the homogeneous Dirichlet problem

$$\begin{aligned} -u''(x) &= 1, \qquad x \in [0,1], \\ u(0) &= 0, \\ u(1) &= 1. \end{aligned}$$

Let $\mathcal{T}_h$ be a triangulation of $[0,1]$, i.e. a set of points $0 = x_0 < x_1 < \ldots < x_N = 1$. Let X_h^1 be the space of continuous, piecewise linear polynomials with respect to $\mathcal{T}_h$, i.e.

$$X_h^1 = \{v \in C^0([0,1]) : v|_{[x_i,x_{i+1}]} \in \mathcal{P}_1 \;\; \forall x_i, x_{i+1} \in \mathcal{T}_h\}.$$

The corresponding Galerkin problem is then: find $u_h \in X_h^1$ such that

$$\begin{aligned} a(u_h, v_h) &= F(v), \quad \forall v_h \in X_h^1, \\ a(u, v) &= \int_0^1 u'(x)v'(x)\,dx, \\ F(v) &= \int_0^1 v(x)\,dx. \end{aligned}$$

This problem is solved by the following Matlab/Python code, where $\mathcal{T}_h$ is set to be 20 equidistant points on $[0, 1]$:

```
from __future__ import division
import numpy as np
import numpy.linalg

n = 20
x = np.linspace(0, 1, n)
A = np.zeros((n, n))
b = np.zeros(n)
h = 1/n

for i in range(n-1):
    A[i:i+2, i:i+2] = A[i:i+2, i:i+2] + np.array([[1, -1], [-1, 1]])/h
    b[i:i+2] = b[i:i+2] + h/2

A[0, 0:2] = [1, 0]  # boundary conditions
A[-1, -2:] = [0, 1]
b[0] = 0
b[-1] = 1
u = numpy.linalg.solve(A, b)
```

a) Modify the code to instead solve the following mixed boundary value problem

$$\begin{aligned} -u''(x) &= 1, \quad x \in [0, 1], \\ u(0) &= 0, \\ u'(1) &= 1. \end{aligned}$$

b) What is the exact solution to this problem? Plot your finite element solution and the exact solution in the same plot.

c) Modify your code to solve the problem using quadratic elements, i.e. where $V_h = X_h^2$, the space of piecewise quadratic polynomials on $\mathcal{T}_h$. Recall that you will require to specify an internal node x_{i+1} on each element $[x_i, x_{i+2}]$. The stiffness matrix A should be constructed from 3×3 sub-blocks of the form

$$\frac{1}{3h}\begin{bmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7 \end{bmatrix}$$

where h is the width of the element. Explain how this form is derived.

3 True or false:

a) The set $\mathcal{S} = \{v \in C^0((0,1)) \mid v(\frac{1}{2}) = 1\}$ is a linear vector space.

b) For $X = H_0^1((0,1))$, $L(v) = \int_0^1 xv\,dx$ is a linear functional.

c) For $X = \mathbb{R}$, $(x,y)_X := |x||y|$ is a valid inner product.

d) The only v in $H^1(\Omega)$ for which $|w|_{H^1(\Omega)}$ (the H^1 semi-norm) is zero is $v = 0$.

4 Consider the fourth-order problem:

$$\begin{aligned} u_{xxxx} &= f \qquad \text{in } \Omega = (0,1), \\ u(0) = u_x(0) = u(1) = u_x(1) &= 0. \end{aligned}$$

This "biharmonic" equation is relevant to, amongst other applications, the bending of beams.

a) Find a symmetric, positive form a over V and a linear form F such that the solution u of the equation satisfies

$$a(u,v) = F(v), \qquad \forall v \in V.$$

b) How should V be defined?