NTNU
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Department of Mathematical Sciences

# Examination paper for
# TMA4220 Numerical Solution of Partial Differential Equations Using Element Methods—SOLUTION

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**Examination time (from–to):** 15:00–19:00

**Permitted examination support material:** C:

- A. Quarteroni: *Numerical Models for Differential Problems, Springer 2014*
- S. Brenner and L. R. Scott: *The Mathematical Theory of Finite Element Methods, Springer 2008*
- *TMA4220 Lecture Notes Fall 2019* (Front page + 229 pages)
- *TMA4220-2019H-AFEM* (25 pages)
- Rottmann: *Matematisk formelsamling*
- Approved calculator

**Other information:**
All answers should be justified and include enough details to make it clear which methods and/or results have been used. All the (sub-)problems are worth 5 points each. The total value is 65 points.

**Language:** English

**Number of pages:** 7

**Number of pages enclosed:** 0

**Checked by:**

Date Signature

**Problem 1**

**a)** We multiply the PDE with $v$ and integrate by parts:

$$\int_\Omega -u_{xx} v\, dx = \int_\Omega f v\, dx$$
$$\int_\Omega u_x v_x\, dx - \int_{\partial\Omega} u_x v\, ds = \int_\Omega f v\, dx$$
$$\int_\Omega u_x v_x\, dx = \int_\Omega f v\, dx + \int_{\partial\Omega_N} \bar{t} v\, ds$$

From this derivation, we see that

$$a(u,v) = \int_\Omega u_x v_x\, dx$$
$$f(v) = \int_\Omega f v\, dx + \int_{\partial\Omega_N} \bar{t} v\, ds$$
$$X = \{u \in H^1(\Omega) \,:\, u = 0 \text{ on } \partial\Omega_D,\, H(u) = 0 \text{ on } \partial\Omega_N\}$$

In our case, $H(u) = u_x(\partial\Omega_N)$, i.e. the value of $u_x$ at most one of the endpoints of $\Omega$. Both endpoints cannot be included because the problem will not have a unique solution in that case.

**b)** Find $u_h \in X_h$ such that we have

$$a(u_h, v_h) = f(v_h) \qquad , \qquad \forall v_h \in X_h$$

Here, $X_h$ is the finite-dimensional subspace of $X$.

**c)** Since $u_h(x) = \sum_{j=1}^N u_j \phi_j(x)$, the elements in $\mathbf{Ax} = \mathbf{b}$ are given by

$$A_{ij} = \int_\Omega \phi_i(x)\phi_j(x)\, dx$$
$$x_i = u_i$$
$$b_i = \int_\Omega f(x)\phi_i(x)\, dx + \delta_{i,N}\bar{t}$$

The last element in $\mathbf{b}$, $b_N$, is due to Neumann conditions, where $\delta_{i,N} = 1$ if endpoint $i$ has Neumann conditions, and otherwise it is zero. The contribution to the load vector equals $\bar{t}$, since $\phi_i(x) = 1$ at the endpoints.

**d)** On the reference interval $[0, 1]$, the basis functions and their derivatives are

$$\phi_0(\xi) = (1-\xi)(1-2\xi) \qquad \phi_0'(\xi) = 4\xi - 3$$
$$\phi_1(\xi) = 4(1-\xi)\xi \qquad \phi_1'(\xi) = 4 - 8\xi$$
$$\phi_2(\xi) = \xi(2\xi - 1) \qquad \phi_2'(\xi) = 4\xi - 1$$

We compute 4 of the 9 entries:

$$\widehat{a}_{11} = \int_0^1 \phi_0'(\xi)\phi_0'(\xi)\,d\xi = \int_0^1 16\xi^2 - 24\xi + 9\,d\xi = \frac{7}{3}$$
$$\widehat{a}_{12} = \int_0^1 \phi_0'(\xi)\phi_1'(\xi)\,d\xi = \int_0^1 -4(8\xi^2 - 10\xi + 3)\,d\xi = -\frac{8}{3}$$
$$\widehat{a}_{13} = \int_0^1 \phi_0'(\xi)\phi_2'(\xi)\,d\xi = \int_0^1 16\xi^2 - 16\xi + 3\,d\xi = \frac{1}{3}$$
$$\widehat{a}_{22} = \int_0^1 \phi_1'(\xi)\phi_1'(\xi)\,d\xi = \int_0^1 16(4\xi^2 - 4\xi + 1)\,d\xi = \frac{1}{3}$$

The reference element is symmetric with respect to the main and minor diagonals, and this yields the relations $\widehat{a}_{33} = \widehat{a}_{11}$, $\widehat{a}_{13} = \widehat{a}_{31}$ and $\widehat{a}_{12} = \widehat{a}_{21} = \widehat{a}_{23} = \widehat{a}_{32}$. Thus, the reference element is defined as

$$\widehat{\mathbf{A}} = \frac{1}{3}\begin{bmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7 \end{bmatrix}$$

**e)** The domain is $\Omega = [0, 1]$, and we have three elements of equal length $h = 1/3$. Thus, the global scaling factor of the matrix $\widehat{\mathbf{A}}$, is $1/(1/3) = 3$, so each element matrix $\mathbf{A}_K$ is multiplied by 3. The first row and column are removed due to Dirichlet conditions. Since the $C_0$-continuity causes overlapping on the last entries of the element matrices, we get

$$\mathbf{A} = \begin{bmatrix} 16 & -8 & 0 & 0 & 0 & 0 \\ -8 & 14 & -8 & 1 & 0 & 0 \\ 0 & -8 & 16 & -8 & 0 & 0 \\ 0 & 1 & -8 & 14 & -8 & 1 \\ 0 & 0 & 0 & -8 & -16 & -8 \\ 0 & 0 & 0 & 1 & -8 & 7 \end{bmatrix}$$

## Problem 2

**a)** A finite element $(K, \mathcal{P}, \mathcal{N})$ satisfies three criterions:

1. $K \subseteq \mathbb{R}^n$ is a bounded, nonempty and closed set with a boundary $\partial K$ which is piecewise continuous.
2. $\mathcal{P}$ is a finite-dimensional space of shape functions on $K$.
3. $\mathcal{N} = \{N_i\}_{i=1}^k$ is a basis for $\mathcal{P}'$ (the dual space of $\mathcal{P}$).

If an element is compatible (conforming), then the approximate space $X_h$ is a true subset of the variational space $X$. This happens when we have continuity between the elements.

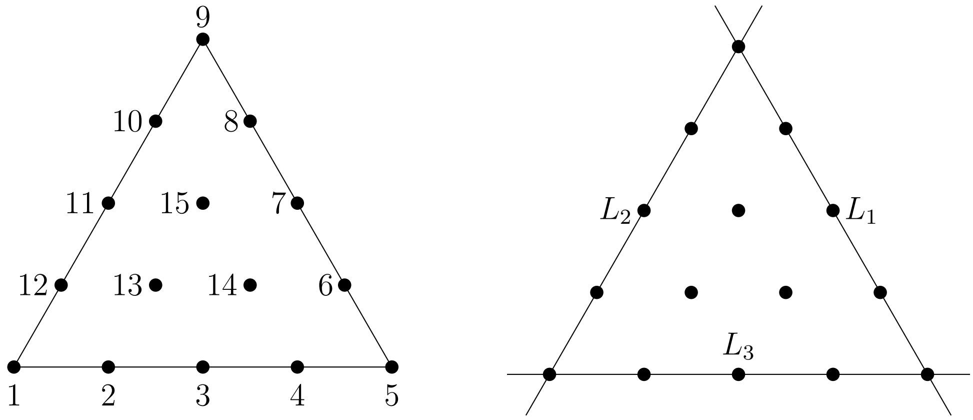


Figure 1: The quartic Langrangian element $\mathcal{P}_4$.

**b)** Let $P \in \mathcal{P}_4$ be an element vanishing on all the nodes. We have three non-trivial linear functions $L_1$, $L_2$ and $L_3$ defining the edges of the triangle. Since $P$ vanishes at the edges and vertex nodes $\{z_i\}_{i=1}^{12}$, there is a linear polynomial $Q$ such that $P = QL_1L_2L_3$. Thus the degrees of freedom span $\mathcal{P}_4'$.
Since $L_1$, $L_2$ and $L_3$ are not zero on the interior points $\{z_i\}_{i=13}^{15}$, and none of these points are co-linear (lying on the same line), $Q$ will vanish on them. This can only be achieved if $Q \equiv 0$, so $P \equiv 0$, and $\mathcal{P}_4$ is compatible.
Any 4th order polynomial along $L_1$, $L_2$ and $L_3$ is uniquely determined by the 5 nodal degrees on the corresponding edge. Since neighbouring elements share the nodal degrees on a common edge, the $C^0$-continuity is satisfied. Hence, the finite-dimensional space $X_h$ is a subspace of $X$, i.e. $X_h \subset X$.

**c)** If a coordinate mapping $\mathcal{F}$ is affine, it can be expressed as $\mathcal{F}(\mathbf{x}) = \mathbf{A}\mathbf{x} + \mathbf{b}$, where $\mathbf{A}$ is an invertible matrix. This mapping preserves polynomials, and the Jacobian matrix is also constant.

**d)** On the bilinear Lagrangian element, we have 4 basis functions:

$$\varphi_1 = \frac{1}{4}(1-\xi)(1-\eta) \qquad \varphi_2 = \frac{1}{4}(1+\xi)(1-\eta)$$
$$\varphi_3 = \frac{1}{4}(1+\xi)(1+\eta) \qquad \varphi_4 = \frac{1}{4}(1-\xi)(1+\eta)$$

They form a partition of unity, $\sum_{i=1}^{4} \varphi_1 = 1$. We assume that the shape functions for the transfer element are on the form

$$\widehat{\varphi}_1 = \varphi_1 - \alpha_1\varphi_5 \qquad \widehat{\varphi}_2 = \varphi_2 - \alpha_2\varphi_5$$
$$\widehat{\varphi}_3 = \varphi_3 - \alpha_3\varphi_5 \qquad \widehat{\varphi}_4 = \varphi_4 - \alpha_4\varphi_5$$
$$\widehat{\varphi}_5 = \varphi_5$$

Since they should form a partition of unity too, and $\varphi_5$ is nontrivial, direct summation yields the following criterion:

$$\sum_{i=1}^{5} \widehat{\varphi}_i = 1$$
$$\varphi_5 + \sum_{i=1}^{4} \varphi_i - \left(\sum_{i=1}^{4} \alpha_i\right) \varphi_5 = 1$$
$$\left(1 - \sum_{i=1}^{4} \alpha_i\right) \varphi_5 = 0$$
$$\sum_{i=1}^{4} \alpha_i = 1$$

Furthermore, $\{\widehat{\varphi}_i\}_{i=1}^5$ must satisfy $\widehat{\varphi}_i(z_j) = \delta_{ij}$ on the nodes. First, we observe that $\phi_1(z_j) = \delta_{1j}$ and $\phi_4(z_j) = \delta_{4j}$ for $j \in [1, 5]$, i.e.,

$$\widehat{\varphi}_1 = \varphi_1 \implies \alpha_1 = 0$$
$$\widehat{\varphi}_4 = \varphi_4 \implies \alpha_4 = 0$$

This implies that $\alpha_2 + \alpha_3 = 1$. The shape function $\phi_5$ is the classic midpoint function for second order Lagrange elements:

$$\widehat{\varphi}_5 = 2\xi\eta(1 - \eta)$$

Both $\varphi_2$ and $\varphi_3$ equal $1/2$ on $z_5$ while $\widehat{\varphi}_5$ equals 1 here, and since $\widehat{\varphi}_2$ and $\widehat{\varphi}_3$ should equal 0 on this node, we get

$$\widehat{\varphi}_2 = \varphi_2 - \frac{1}{2}\varphi_5 \implies \alpha_1 = \frac{1}{2}$$
$$\widehat{\varphi}_3 = \varphi_3 - \frac{1}{2}\varphi_5 \implies \alpha_4 = \frac{1}{2}$$

which fulfils the partition of unity.

## Problem 3

**a)** If $I_X$ is an interpolant, $u$ is a function, and $u_X$ is a FEM-approximation, then the superconvergence property requires the constants $C(u)$ and $\tau \in (0, 1]$ independent of $h$ such that

$$|u_X - I_X u|_{H^1} \leq C(u) h^{p+\tau}$$

It is present when the solution is smooth enough to avoid pollution and the mesh is quasi-uniform.

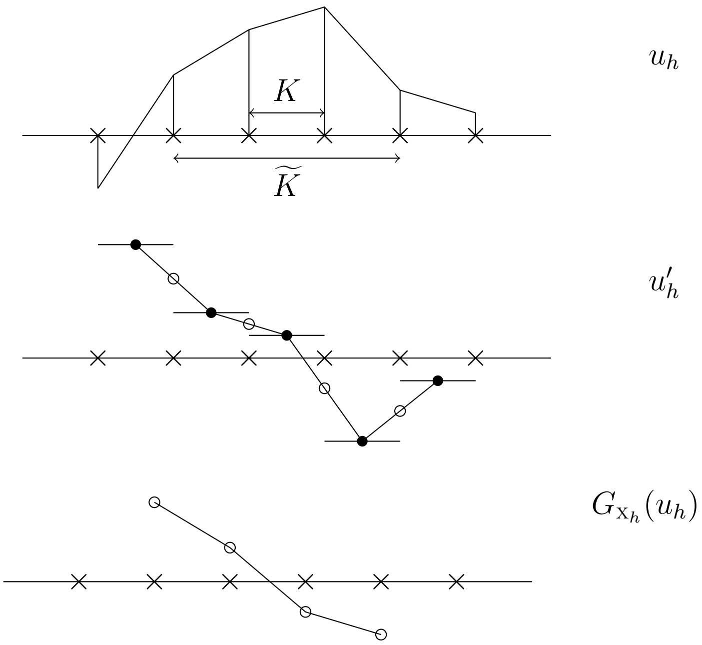


Figure 2: Construction of the recovered gradient in 1D for linear elements.

**b)** The recovery operator $G_{X_h}$ satisfies three properties:

1. Consistency: $G_{X_h}(I_{X_h}u) = I_{X_h}u'$.
2. Localization: $G_{X_h}$ depends only on sampled values on its patch.
3. Boundedness and linearity.

**c)** In 1D, we have a set of nodes $\{x_i\}_{i=0}^N$ forming a non-uniform partition on the interval $[a,b]$, such that $x_0 = a$ and $x_N = b$. Since the elements are linear, their derivatives are constant, so we choose their centroids as sampling points (as the midpoint is the so-called Barlow point for linear elements):

$$z_i = \frac{x_i + x_{i-1}}{2} \qquad , \qquad 1 \le i \le N$$

Thus, we obtain a coordinate set $\{(z_i, u_h'(z_i))\}_{i=1}^N$, and we use them to create a new set of $N-1$ linear functions:

$$s_i(x) = \frac{u_h'(z_i) - u_h'(z_{i-1})}{z_i - z_{i-1}}(x - z_{i-1}) + u_h'(z_{i-1}) \qquad , \qquad 2 \le i \le N$$

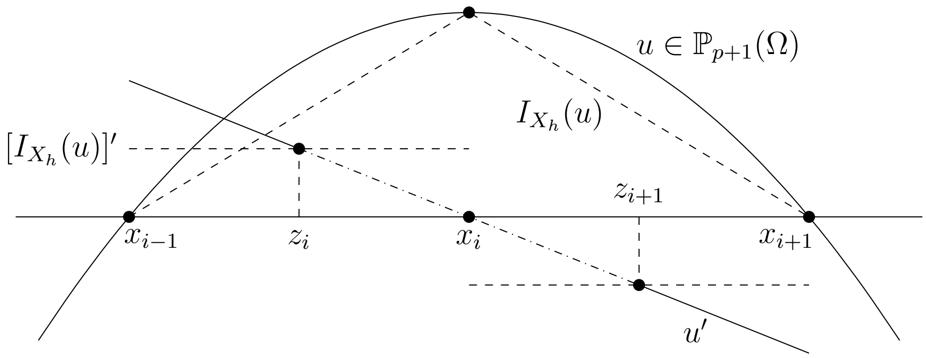


Figure 3: Superconvergent Patch Recovery: Consistency check for the recovery operator, $G_{X_h}(I_{X_h}u) = I_{X_h}u'$.

Next, we evaluate $s_i$ on $x_i$, and this yields a new set of points:

$$\begin{aligned} s_i(x_i) &= \frac{u_h'(z_i) - u_h'(z_{i-1})}{\frac{x_{i+1}+x_i}{2} - \frac{x_i+x_{i-1}}{2}} \left( x_i - \frac{x_{i-1}+x_i}{2} \right) + u_h'(z_{i-1}) \\ &= (u_h'(z_i) - u_h'(z_{i-1})) \frac{x_i - x_{i-1}}{x_{i+1} - x_{i-1}} + u_h'(z_{i-1}) \end{aligned}$$

This set of values, $\{s_i(x_i)\}_{i=2}^{N}$, are now used for defining the $C^0$-continuous piecewise linear function for the recovered gradient:

$$G_{\mathrm{X}_h}(x) = \sum_{i=1}^{N-1} s_i(x_i)\phi_i(x) \qquad , \qquad x \in \left[ \frac{x_0 + x_1}{2}, \frac{x_{N-1} + x_N}{2} \right]$$

At the left and right end, $G_{\mathrm{X}_h}(x)$ is extrapolated to the endpoints.

If $u_h$ and $v_h$ are two FEM-solutions, then $G_{\mathrm{X}_h}(u_h + v_h) = G_{\mathrm{X}_h}(u_h) + G_{\mathrm{X}_h}(v_h)$ because the recovered gradients are a linear sum of functions.
Boundedness of the operator follows from the fact that the sampled points of $u_h'$ are all bounded, and the localization is satisfied because we only use points on the patches.
In order to prove consistency, we see from Figure 3 that we have $u(x) = -(x - x_{i-1})(x - x_{i+1})$, so

$$I_{X_h}(u) = \begin{cases} (x_{i+1} - x_i)(x - x_{i-1}) & , x \leq x_0 \\ -(x_i - x_{i-1})(x - x_{i+1}) & , x > x_0 \end{cases}$$

This implies that

$$[I_{X_h}(u)]' = \begin{cases} x_{i+1} - x_i & , x \leq x_0 \\ -(x_i - x_{i-1}) & , x > x_0 \end{cases}$$

We see that

$$[I_{X_h}(u)]'(z_i) = x_{i+1} - x_i \qquad , \qquad [I_{X_h}(u)]'(z_{i+1}) = -(x_i - x_{i-1})$$

The straight line connecting these points is

$$\begin{aligned} s(x) &= \frac{[I_{X_h}(u)]'(z_{i+1}) - [I_{X_h}(u)]'(z_i)}{z_{i+1} - z_i}(x - z_i) + [I_{X_h}(u)]'(z_i) \\ &= \frac{-(x_i - x_{i-1}) - (x_{i+1} - x_i)}{\frac{x_{i+1}+x_i}{2} - \frac{x_i + x_{i-1}}{2}}\left(x - \frac{x_i + x_{i-1}}{2}\right) + x_{i+1} - x_i \\ &= -2x + x_i + x_{i-1} + x_{i+1} - x_i \\ &= -2x + x_{i-1} + x_{i+1} \end{aligned}$$

By comparing this with the exact derivative, we see that

$$\begin{aligned} u(x) &= -(x - x_{i-1})(x - x_{i+1}) \\ u'(x) &= -2x + x_{i-1} + x_{i+1} \end{aligned}$$

Since the derivatives equal each other, consistency has been proven.

**d)** In error estimation, we seek to compute the exact error:

$$\|u - u_h\|_{X_h}$$

Since the exact solution $u$ is not available in general, we can use the recovered gradient to split this exact error in two pairs:

$$\|\nabla(u - u_h)\|_{L^2} = \|\nabla u - G_{X_h}(u_h)\|_{L^2} + \|G_{X_h}(u_h) - \nabla u_h\|_{L^2}$$

Since the recovery operator is constructed properly, we have

$$\|\nabla(u - u_h)\|_{L^2} \approx \|G_{X_h}(u_h) - \nabla u_h\|_{L^2}$$

We can use $\|G_{X_h}(u_h) - \nabla u_h\|_{L^2}$ to estimate the numerical error.