NTNU
Norwegian University of
Science and Technology

Department of Mathematical Sciences

Examination paper for
TMA4220 Numerical Solution of Partial Differential Equations Using Element Methods

Academic contact during examination: Trond Kvamsdal

Phone: 93058702

Examination date: 2nd of December 2019

Examination time (from–to): 15:00–19:00

Permitted examination support material: C:

- A. Quarteroni: *Numerical Models for Differential Problems, Springer 2014*
- S. Brenner and L. R. Scott: *The Mathematical Theory of Finite Element Methods, Springer 2008*
- *TMA4220 Lecture Notes Fall 2019* (Front page + 229 pages)
- *TMA4220-2019H-AFEM* (25 pages)
- Rottmann: *Matematisk formelsamling*
- Approved calculator

Other information:
All answers should be justified and include enough details to make it clear which methods and/or results have been used. All the (sub-)problems are worth 5 points each. The total value is 65 points.

Language: English

Number of pages: ??

Number of pages enclosed: 0

Checked by:

Informasjon om trykking av eksamensoppgave
Originalen er:
1-sidig ☐ **2-sidig** ☒
sort/hvit ☒ **farger** ☐
skal ha flervalgskjema ☐

Date Signature

Problem 1 Consider the one-dimensional Poisson equation

$$-u_{xx} = f \quad \text{in} \quad \Omega \tag{1}$$

with the boundary conditions

$$u = 0 \quad \text{on} \quad \partial\Omega_D \tag{2}$$
$$H(u) = \bar{t} \quad \text{on} \quad \partial\Omega_N. \tag{3}$$

where u is the unknown solution, f the applied loading and $\bar{t}$ are prescribed Neumann boundary conditions for the given Poisson problem defined on the interval $\Omega \in R^1$. The differential operator $H(u)$ is the so called "Neumann operator".

a) Use Galerkin's method and establish the weak formulation corresponding to the problem (**??**)-(**??**) on the form: Find $u \in X$ such that

$$a(u, v) = l(v) \qquad \forall v \in X \tag{4}$$

In particular, identify X, a and l for this problem. Identify also the expression for the "Neumann operator" $H(u)$ related to the Neumann boundary conditions.

b) Assume that we choose the finite element method to solve (**??**) numerically. Formulate the corresponding finite element variational formulation.

c) Establish the linear algebraic system of equations in the form: $\mathbf{Ax} = \mathbf{b}$, where $\mathbf{A}$ is the system coefficient matrix (also denoted "stiffness matrix"), $\mathbf{x}$ is the vector of unknown nodal coefficients and $\mathbf{b}$ is the right hand side due to applied loading and prescribed boundary conditions given in (**??**)-(**??**).

d) Introduce quadratic Lagrange nodal basis functions to express the finite element solution u_h and the corresponding test function v_h for the finite element variational problem above. Compute the coefficient matrix ("element stiffness matrix") $\mathbf{A_e}$ for one quadratic Lagrange 1D-element (line element with three nodes).

e) Compute the system coefficient matrix $\mathbf{A}$ for the above finite element Poisson problem where $\Omega = (0, 1)$, $\partial\Omega_D = \{x = 0\}$ and $\partial\Omega_N = \{x = 1\}$ that is discretized with three quadratic Lagrange 1D-elements of equal length.

Problem 2

a) What characterize a finite element? What is a compatible finite element?

b) Define a compatible quartic (i.e. 4^{th} order) triangular Lagrange type finite element, i.e. show the nodal locations. Verify that your quartic triangular finite element fulfils the conditions necessary to construct a compatible finite element space.

c) Define "affine coordinate mapping". Which property inherent in affine coordinate mappings is consider to be of special interest related to the finite element method.

d) Define the nodal basis functions for the 5 node “transfer element” shown in Figure 1, such that it is compatible with neighbouring standard 9 node quadratic Lagrange quadrilateral element along the edge with the midpoint node, and compatible with a neighbouring 4 node linear Lagrange quadrilateral element along each of the other edges.

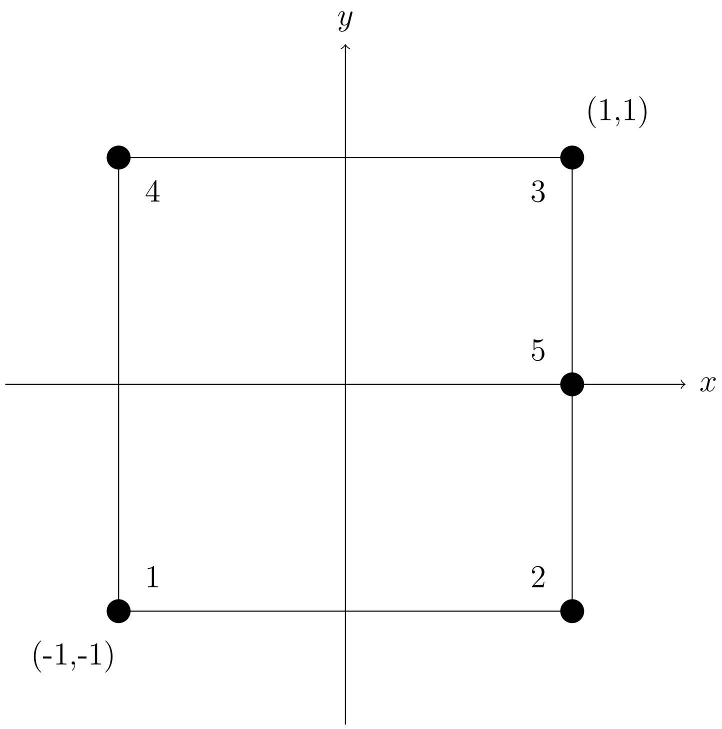

Figure 1: "Transfer element” with 5 nodes

Problem 3 Here we again consider a one-dimensional Poisson problem as described in Problem 1, i.e., Equation (1)–(3) with the weak formulation as given in Equation (4) to be solved with a corresponding compatible finite element problem using the finite dimensional space X_h.

a) What is the superconvergence property? What is the conditions for it to be present?

b) State the three conditions for the recovery operator G_{X_h} that is sufficient to guarantee that $G_{\mathrm{X}_h}(I_{\mathrm{X}_h}u)$ is a good approximation to the true gradient ∇u. Here I_{X_h} is the interpolant operator on the finite element space X_h.

c) Construct a recovery operator G_{X_h} that fulfills the three conditions found in 3(b) for one-dimensional linear finite elements. Check that your recovery operator is *consistent*.

d) Describe how you can utilise the recovered gradient $G_{\mathrm{X}_h}(u_h)$ to estimate the error in the numerically computed finite element solution $u_h \in X_h$.