NTNU
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Department of Mathematical Sciences

Examination paper for
TMA4220 Numerical Solution of Partial Differential Equations Using Element Methods

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Problem 1 Consider the Poisson equation

$$-\nabla^2 u = f \quad \text{in } \Omega,$$

on the domain$\Omega \ = \{x \in \mathbb{R}^3 : 1 < |x| < 2\}$, where $|\cdot|$ is the Euclidean norm. We impose homogeneous Dirichlet boundary conditions $u = 0$ on $\partial\Omega$.

a) Establish a weak formulation

$$\text{find } u \in V \quad \text{such that} \quad a(u, v) = F(v), \quad \forall v \in V$$

where you should define V and give expressions for a and F.

b) Use the Lax-Milgram theorem to prove that the above problem has a unique solution.

For the rest of this problem, we suppose that $f = -2$ is a constant. We deduce that the solution u will be a function of distance from the origin only, i.e. $u = u(r)$.

c) By transforming to spherical coordinates (see the appendix), show that the bilinear form becomes

$$a(u, v) = 4\pi \int_1^2 r^2 \frac{\partial u}{\partial r} \frac{\partial v}{\partial r} dr,$$

and find an expression for $F(v)$.

We wish to find a numerical solution using linear finite elements on an equidistant grid of points $r_0, \ldots, r_M$.

d) Find the elemental stiffness matrix and elemental load vector for the element $K = [r_{k-1}, r_k]$, where $r_k = 1 + h \cdot k$.

e) Explain how your above answer can be combined to give a linear system

$$A_h u_h = F_h$$

for the approximation solution u_h (you do not need to calculate A_h and F_h explicitly).

Problem 2 Consider the equation

$$-0.05u_{xx} + 2u_x = 0, \quad u(0) = 0,\ u(1) = 1$$

a) We find an approximate solution u_h to the above problem using linear finite elements on a grid with nodal points $x_n, n = 0, \dots, M$, not assumed to be equidistant. Find an expression relating u_{n+1}, u_n and u_{n-1}, where $u_n = u_h(x_n)$ is the approximate solution at the node x_n (The coefficients of the relation will involve the unknown element sizes h_n).

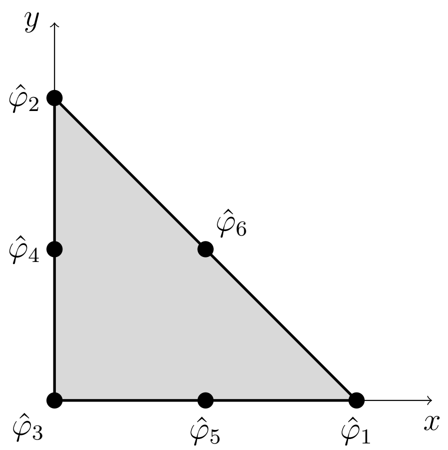
b) Using the error estimate

$$||u - u_h||_{H^1(\Omega)} \leq C \left(\sum_i h_i^2 |u|^2_{H^2([x_i, x_{i+1}])} \right)^{\frac{1}{2}},$$

find a relation satisfied by the points x_n such that the error is approximately equidistributed over the elements (do not solve for x_n!)

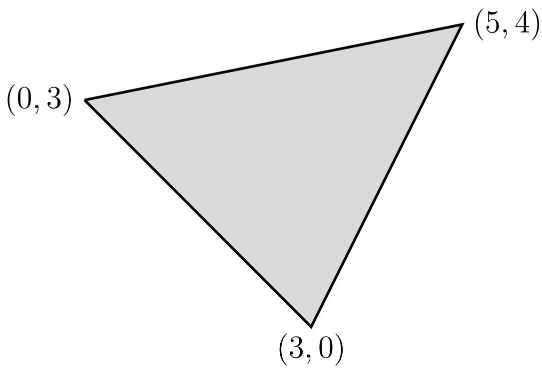
c) For too large stepsizes, the approximate solution obtained by the linear finite element method can oscillate rapidly, even though the true solution is monotone increasing. Explain how the method can be modified to counteract this behaviour.

Problem 3 We consider quadratic nodal basis functions, first given on the reference element $\hat{K}$ with vertices at $(1,0), (0,1)$ and $(0,0)$ and numbered as follows:



a) Give expressions for the basis functions $\hat{\varphi}_1, \hat{\varphi}_5$ in barycentric coordinates.

b) We wish to evaluate the elemental mass matrix $\int_K \varphi_i \varphi_j$ on the element K with vertices $x_1 = (3, 0), x_2 = (5, 4), x_3 = (0, 3)$, where φ_i are the quadratic nodal basis functions on K. (Number these so that for $i = 1, 2, 3$, the φ_i are nonzero at x_i, and for $i = 4, 5, 6$ follow the pattern from the reference element)



Use the one-point Gaussian quadrature rule based on evaluation at the barycentre to find approximations for the following integrals:

$$\int_K \varphi_1 \varphi_1, \quad \int_K \varphi_1 \varphi_5$$

Problem 4 Suppose we solve the one-dimensional heat equation

$$u_{xx} + u = u_t, \quad u(0, t) = u(1, t) = 0, \quad u(x, 0) = g(x)$$

using linear finite elements on a uniform subdivision of the interval $[0, 1]$, with element diameter h. In this problem, we will study the stability of the forward Euler method.

a) First solve the generalized eigenvalue problem

$$A_h v = \lambda_h M_h v,$$

where A_h and M_h are the stiffness and mass matrices.

Hint: a matrix is Toeplitz if its entries are constant along diagonals, i.e. $a_{i,j} = a_{i+1,j+1}$ for all i, j. Of particular interest are matrices that are tridiagonal, symmetric and Toeplitz (TST):

1. All $n \times n$ TST matrices have the same eigenvectors

2. The eigenvalues of an $n \times n$ TST matrix with diagonal entries a and off-diagonal entries b are given by

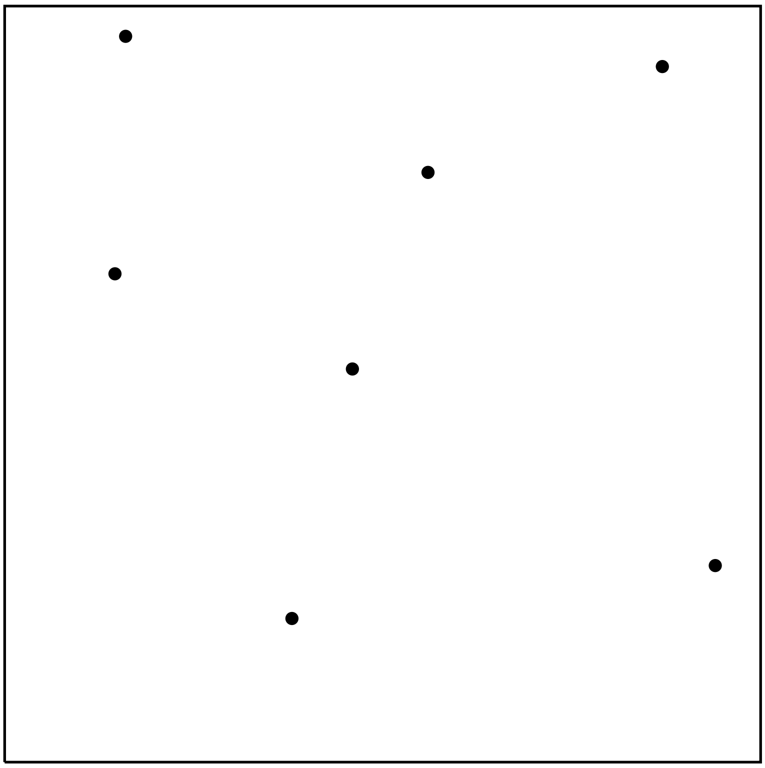
$$a + 2b\cos(\frac{k\pi}{n+1}), \quad k = 1, \dots, n$$

b) Use the above to derive a formula for the maximum step-size k in the time direction for which the Euler method is stable, as a function of h.

Problem 5

a) What is a Delaunay triangulation of a set of points $x_1, \dots, x_M \in \mathbb{R}^2$?

b) Draw the Delaunay triangulation of the following set of points (you can either draw on the exam sheet or copy the figure to a separate sheet):



Appendix: spherical coordinates

$$\nabla g = \left(\frac{\partial g}{\partial r}, \frac{1}{r}\frac{\partial g}{\partial \theta}, \frac{1}{r \sin\theta}\frac{\partial g}{\partial \phi} \right)^T$$

$$\nabla^2 g = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 \frac{\partial g}{\partial r}) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}(\sin\theta \frac{\partial g}{\partial \theta}) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 g}{\partial \phi^2}$$

$$dV = r^2 \sin\theta \, dr \, d\theta \, d\phi$$

where $\theta \in [0, \pi)$ is the polar angle and $\phi \in [0, 2\pi)$ is the azimuthal angle.