NTNU
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Department of Mathematical Sciences

Examination paper for
TMA4220 Numerical Solution of Partial Differential Equations Using Element Methods

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Originalen er:
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Problem 1 Consider the following differential equation in strong form

$$
\begin{aligned}
-\frac{\mathrm{d}}{\mathrm{d}x}\left(\beta(x)\frac{\mathrm{d}u}{\mathrm{d}x}\right) &= f(x), \quad x \in [0,1] \\
u(0) &= 0 \\
u(1) &= 0
\end{aligned}
$$

a) The weak form of the problem is find $u \in V$ such that

$$a(u,v) = F(v), \quad \forall v \in V$$

Show that

$$a(u,v) = \int_0^1 \beta(x)\frac{\mathrm{d}u}{\mathrm{d}x}\frac{\mathrm{d}v}{\mathrm{d}x}\mathrm{d}x \tag{1}$$

What is $F(v)$? What is V?

For the rest of the problem, assume $\beta(x) = x + 1$

b) The Lax-Milgram theorem ensures a unique solution to the weak problem if the bilinear form $a(u,v)$ is *coercive* and *continuous* (see appendix). Show that the bilinear form (1) satisifes this when $\beta(x) = x + 1$.

c) We discretize this using the linear hat functions $V_h = X_h^1 = \text{span}\{\varphi_1, \varphi_2, ..., \varphi_n\}$, where $V_h \subset V$ is our subspace. The Galerkin method states that we should find $u_h \in V_h$ such that

$$a(u_h, v_h) = F(v_h), \quad \forall v_h \in V_h$$

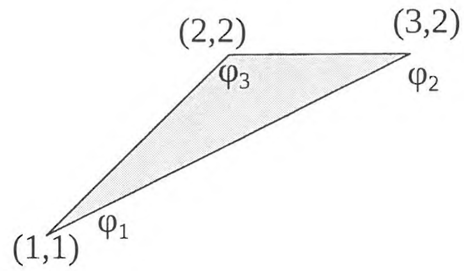
Show how we use this to get to the fully discrete linear system of equations

$$A\mathbf{u} = \mathbf{b}$$

d) Assume that we discretize X_h^1 using a uniform grid, i.e. the nodes $x = \{0, h, 2h, 3h, ..., 1\}$ for some element size h. Show that the matrix A is tridiagonal and

$$
\begin{aligned}
a_{i,i-1} &= -\frac{1}{2h}\left(2 + h(2i-1)\right) \\
a_{i,i} &= \frac{2}{h}\left(1 + ih\right) \\
a_{i,i+1} &= -\frac{1}{2h}\left(2 + h(2i+1)\right)
\end{aligned}
$$

Problem 2 Consider the following triangle and the linear hat functions X_h^1 on this triangle



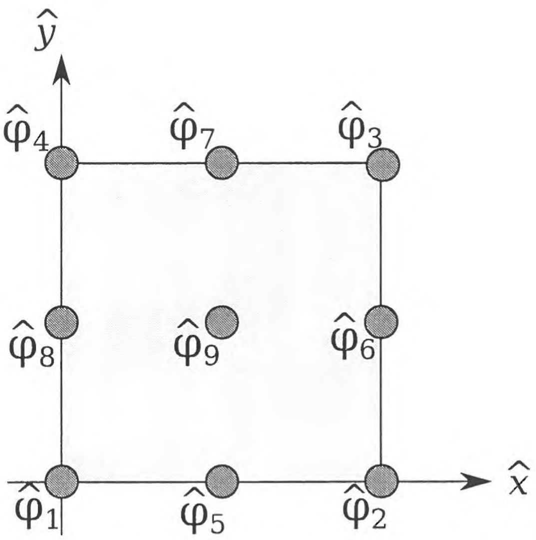
a) What are the linear basis functions $\varphi_1(x, y)$, $\varphi_2(x, y)$ and $\varphi_3(x, y)$?

b) The following `matlab` program will generate some mesh with nodes and triangles connecting these nodes.

```
[points, triangles] = getMesh(100);
numb_triangles = size(triangles,2);
for k=1:numb_triangles
  I = triangles(k,:);
  x = points(I,1);
  y = points(I,2);
  % insert your code here
end
```

The `for`-loop iterates over all elements (triangles) and stores the 3 corners in the `x` and `y` variable, both vectors of size 3. How would you compute the basis functions $\varphi_i(x, y) = a_i x + b_i y + c_i$ $(i = \{1, 2, 3\})$ for a general triangle here? Describe your method using either `matlab` code, or pseudo-code.

Problem 3 Consider the reference element $(\hat{x}, \hat{y}) \in [0,1]^2$ for biquadratic quadrilatherals Y_h^2 given by



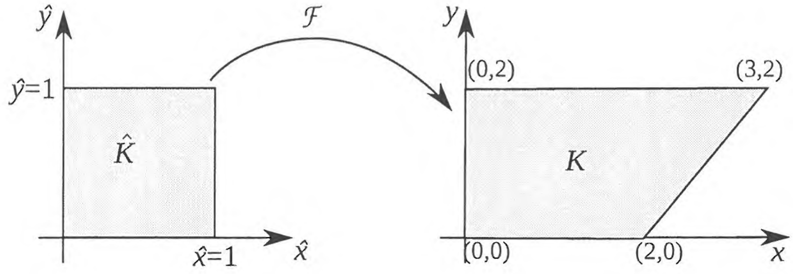
a) Denote by $\hat{\nabla} = \begin{bmatrix} \frac{\partial}{\partial \hat{x}} \\ \frac{\partial}{\partial \hat{y}} \end{bmatrix}$ the derivatives with respect to the reference coordinates. What are the basis functions

(i) $\hat{\varphi}_1(\hat{x}, \hat{y})$
(ii) $\hat{\varphi}_5(\hat{x}, \hat{y})$
(iii) $\hat{\varphi}_9(\hat{x}, \hat{y})$
(iv) $\hat{\nabla}\hat{\varphi}_1(\hat{x}, \hat{y})$
(v) $\hat{\nabla}\hat{\varphi}_9(\hat{x}, \hat{y})$

b) What is the bilinear mapping

$$\begin{aligned} x(\hat{x}, \hat{y}) &= a_1\hat{x}\hat{y} + a_2\hat{x} + a_3\hat{y} + a_4 \\ y(\hat{x}, \hat{y}) &= b_1\hat{x}\hat{y} + b_2\hat{x} + b_3\hat{y} + b_4 \end{aligned}$$

from the reference coordinates $(\hat{x}, \hat{y})$ to the physical coordinates (x, y) on the following quadrilatheral



c) What is the jacobian J of this mapping

$$J = \begin{bmatrix} \frac{\partial x}{\partial \hat{x}} & \frac{\partial x}{\partial \hat{y}} \\ \frac{\partial y}{\partial \hat{x}} & \frac{\partial y}{\partial \hat{y}} \end{bmatrix}$$

d) The physical derivatives $\nabla = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{bmatrix}$ can be expressed as

$$\nabla \varphi = G \hat{\nabla} \hat{\varphi}$$

For some matrix $G \in \mathbb{R}^{2\times 2}$. What is the matrix G? Use this to compute $\nabla \varphi_9$ at the center right edge $(\hat{x}, \hat{y}) = (1, \frac{1}{2})$.

Problem 4 In this task, you are to derive an a priori error estimate for the 1D linear hat functions.

Let $v \in H^2(I)$ on some interval $I = [a, b] \subset \mathbb{R}$. Let $\Pi_h^1 v \in X_h^1$ be the linear interpolant on some nodal mesh given by X_h^1.

a) Let $I = [0, 2\pi]$ and X_h^1 be defined on the uniform mesh with nodes $x_i = \left\{0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi\right\}$. Draw the interpolant $\Pi_h^1 v(x)$ for $v(x) = \sin(x)$.

b) Show that the interpolation error is

$$|v - \Pi_h^1 v|_{H^1(I)} \leq Ch|v|_{H^2(I)}$$

c) Show Ceas Lemma, i.e. for a *coercive*, *continuous* bilinear form $a(\cdot,\cdot) : H^1(I) \times H^1(I) \to \mathbb{R}$, we have

$$\|u - u_h\|_{H^1(I)} \leq \frac{M}{\alpha} \inf_{w_h \in H^1(I)} \|u - w_h\|_{H^1(I)}$$

Use this to prove that the approximation error for the finite element method is bounded by

$$\|u - u_h\|_{H^1(I)} \leq \frac{M}{\alpha} Ch|u|_{H^2(I)}$$

A bilinear form $a(\cdot,\cdot) : V \times V \to \mathbb{R}$ is said to be

Coercive if there exist some $\alpha > 0$:

$$a(v,v) > \alpha \|v\|_V^2$$

Continuous if there exist some $M > 0$:

$$|a(u,v)| \leq M\|u\|_V \|v\|_V$$