Løsningsforslag

TMA4220

Endelige elementmetoden

(skriflig eksamen

21.desember 2016)

#1

a) $\int_0^1 (\beta u_x)_x \, v \, dx = \int_0^1 f v \, dx$

$a(u,v) = F(v) \qquad V = H^1$

b) Poincare

$\int \beta v_x v_x > \min_{x\in[0,1]} \beta \int v_x^2 \geq C^* |v|_{H^1} \geq C \|v\|_{H^1}$

Cauchy-Schwartz

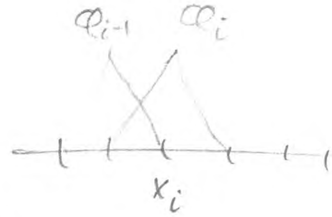
$\int \beta v_x u_x < \max \beta \int v_x u_x \, dx \leq 2 |v|_{H^1} |u|_{H^1}$

$\leq 2 \|v\|_{H^1} \|u\|_{H^1}$ Norm Def.

c) $a(\sum u_i \varphi_i, \varphi_j) = F(\varphi_j)$

$\Rightarrow Au = b$

d) $a_{i,i-1} = \int_{x_{i-1}}^{x_i} -\frac{1}{h^2}(1+x)\,dx$



$= -\frac{1}{h} - \frac{1}{h^2}\left[\frac{1}{2}x^2\right]_{x_{i-1}}^{x_i}$

$= -\frac{1}{h}\left(1 + \frac{h}{2}(2i-1)\right)$

$\cancel{= -\frac{1}{h}\left(i - \frac{1}{2}\right)}$

$x_i = ih$

$(x_i - x_{i-1})(x_i + x_{i-1})$

$h \qquad (ih + ih - h)$

$a_{ii} = \int_{x_{i-1}}^{x_{i+1}} \frac{1}{h^2}(1+x)\,dx = \frac{1}{h^2}\left[x + \frac{1}{2}x^2\right]_{x_{i-1}}^{x_{i+1}}$

$= \frac{1}{h^2}\left(2h + \frac{1}{2}(ih + \cancel{h} + ih - \cancel{h}) \cdot 2h\right)$

$= \frac{1}{h}(2 + 2ih) = \frac{2}{h}(1 + ih)$

$a_{i,i+1} = \int_{x_i}^{x_{i+1}} -\frac{1}{h^2}(1+x)\,dx$

$= -\frac{1}{h^2}\left[x + \frac{1}{2}x^2\right]_{x_i}^{x_{i+1}} = -\frac{1}{h^2}\left(h + \frac{1}{2}((i+1)h + ih) \cdot h\right) = -\frac{1}{2h}(2 + h(2i+1))$

$= -\frac{1}{h}\left(1 + ih + \frac{1}{2}h\right) \quad \cancel{= -\frac{1}{h}\frac{1}{2}}$

#2

a)

$$\underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 3 & 2 & 1 \\ 2 & 2 & 1 \end{bmatrix}}_{\text{invert}} \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\varphi_i = a_i x + b_i y + c_i$$

$\Rightarrow$

$$\varphi_1 = 2 - y$$
$$\varphi_2 = x - y$$
$$\varphi_3 = -x + 2y - 1$$

b) K = [x , y, ones(3,1)]

abc = inv(K)

#3

a)

$$\hat{\varphi}_1 = (\hat{x} - \tfrac{1}{2})(\hat{x} - 1)(\hat{y} - \tfrac{1}{2})(\hat{y} - 1) \cdot 4$$

$$\hat{\varphi}_5 = \hat{x}(1 - \hat{x})(\hat{y} - \tfrac{1}{2})(\hat{y} - 1) \cdot 8$$

$$\hat{\varphi}_9 = \hat{x}\hat{y}(1 - \hat{x})(1 - \hat{y}) \cdot 16$$

$$\hat{\nabla} \hat{\varphi}_1 = 4 \begin{bmatrix} (2\hat{x} - \tfrac{3}{2})(\hat{y} - \tfrac{1}{2})(\hat{y} - 1) \\ (\hat{x} - \tfrac{1}{2})(\hat{x} - 1)(2\hat{y} - \tfrac{3}{2}) \end{bmatrix}$$

$$\hat{\nabla} \hat{\varphi}_9 = 16 \begin{bmatrix} (1 - 2\hat{x})\hat{y}(1 - \hat{y}) \\ \hat{x}(1 - \hat{x})(1 - 2\hat{y}) \end{bmatrix}$$

b) $y = 2\hat{y}$

$x = 2\hat{x} + \hat{y}\hat{x}$

c) $J = \begin{bmatrix} 2+\hat{y} & \hat{x} \\ 0 & 2 \end{bmatrix}$

d) $G = J^{-T}$ (chain-rule)

~~Here:~~ $\cancel{J^{-1} = \frac{1}{4}\begin{bmatrix} 2 & -1 \\ 0 & 2 \end{bmatrix}}$

$\cancel{G = \frac{1}{4}\begin{bmatrix} 2 & 0 \\ -1 & 2 \end{bmatrix}}$

e) $\cancel{\hat{\nabla}\hat{\varphi}_1(1, \tfrac{1}{2}) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}}$

$\hat{\nabla}\hat{\varphi}_9(1, \tfrac{1}{2}) = \begin{bmatrix} -\frac{1}{4} \\ 0 \end{bmatrix} 16 = -4 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

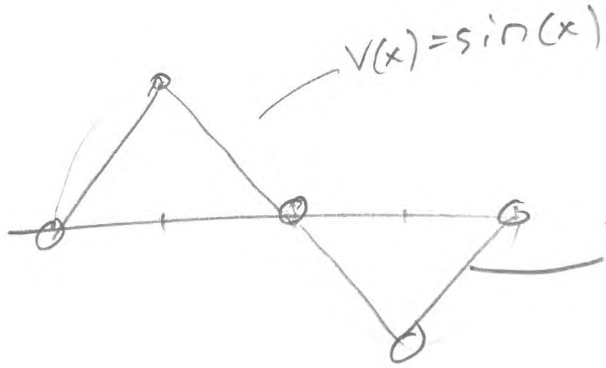
$G = J^{-T}(1, 1/2) = \begin{bmatrix} 5/2 & 1 \\ 0 & 2 \end{bmatrix}^{-1} = \frac{1}{5}\begin{bmatrix} 2 & -1 \\ 0 & 5/2 \end{bmatrix}^{T}$

$\nabla\varphi_9 = G\,\hat{\nabla}\hat{\varphi}_9 = -\frac{4}{5}\begin{bmatrix} 2 & 0 \\ -1 & 5/2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$= +\frac{4}{5}\begin{bmatrix} -2 \\ 1 \end{bmatrix}$

#4

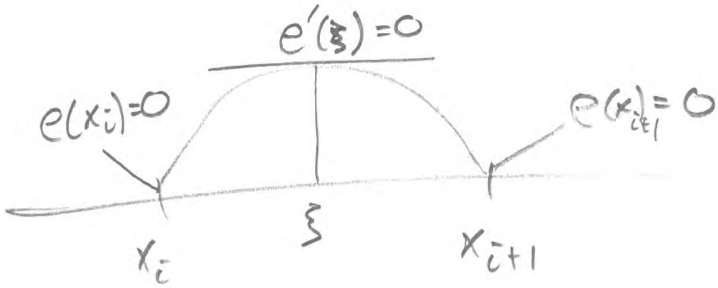
a)



$$\Pi_h^1 v(x) = \sum_{i=1}^{5} v(x_i)\,\varphi_i(x)$$

b) error: $e(x) = v(x) - \Pi_h^1 v(x)$

$$e''(x) = v''(x) \quad \text{since } \left(\Pi_h^1 v(x)\right)'' = 0$$



Rolles Thm:
some $\xi \in [x_i, x_{i+1}]$ exist s.t.
$$e'(\xi) = 0$$

$$|e|^2_{H^1(K)} = \int_{x_i}^{x_{i+1}} (e'(x))\, dx = \int$$

$$\int_\xi^x e''(s)\,ds = e'(x) \qquad |e'(x)| < \int_{x_i}^{x_{i+1}} |e''(s)|\,ds$$

$$\int_{x_i}^{x_{i+1}} |e'(x)|^2 dx < h \cdot \cancel{\int_{x_i}^{x_{i+1}}} \cdot \max |e'(x)|^2$$

$$< h \cdot \left(\int_{x_i}^{x_{i+1}} |e''(s)\,ds|\right)^2$$

$$< h \int_{x_i}^{x_{i+1}} |e''(s)|^2 ds$$

$$= h\,|e|^2_{H^2(K)}$$

$$|e|^2_{H^1(I)} = \sum_{K \in I} |e|^2_{H^1(K}$$

c) $a(u-u_h, u-u_h)$

$= a(u-u_h, u-w_h+w_h-u_h)$ $\in V$

$= a(u-u_h, u-w_h) + \underbrace{a(u-u_h, w_h-u_h)}_{=0}$

Know $a(u-u_h, u-u_h) > \alpha \|u-u_h\|_{H^1}^2$

$a(u-u_h, u-w_h) \leq M\|u-u_h\|_{H^1} \cdot \|u-w_h\|$

$\Rightarrow \alpha\|u-u_h\|_{H^1}^{\cancel{2}} \leq M \cancel{\|u-u_h\|_{H^1}} \|u-w_h\|_{H^1}$

$\square$

d) $\|u-u_h\|_{H^1(I)} \leq \frac{M}{\alpha}\|u-w_h\|_{H^1}$

choose w_h to be the interpolant