Solution to exam, TMA4220

December 10, 2015

Max number of points: 135

Problem 1

a) (10 p)
$V = H^1(\Omega)$,

$$a(u,v) = \int_\Omega \mu \nabla u \cdot \nabla v + b \cdot \nabla u v + \sigma u v \, dx, \qquad F(v) = \int_\Omega f v \, dx + \mu \int_{\partial\Omega} v h \, dx.$$

b) (10 p)
The BVP is

$$\begin{cases} b \cdot \nabla u = 2 & \text{for } x \in \Omega \\ u(x) = 0 & \text{for } x \in \Gamma_- \end{cases}$$

where $\Gamma_- = \{(x_1, x_2) \in \partial\Omega \ : \ x_1 = 0 \text{ or } x_2 = 0\}$.

Characteristics solve $\dot{X}(s) = b$, $X(0) = \bar{X}$, so $X(s) = \bar{X} + sb$. Given $x \in \Omega$, we have $X(s) = x$ when $s = \min(x_1, x_2)$ and $\bar{X} = x - \min(x_1, x_2)b$. Along this characteristic, we have

$$\frac{d}{ds} u(X(s)) = 2 \quad \Rightarrow \quad u(X(s)) = 2s + u(\bar{X}) = 2s.$$

Thus, $u(x) = 2\min(x_1, x_2)$.

c) (5 p)
We need $h \leqslant \frac{\mu}{|b|} = \frac{1}{\sqrt{2}\cdot 100}$.

Problem 2

a) (10 p)
The weak formulation is

$$\text{find } u(t) \in V \text{ such that } \left(\frac{\partial u}{\partial t}(t),\ v\right) + a(u,v) = 0 \qquad \text{for all } v \in V$$

where $V := H_0^1(\Omega)$ and a is the bilinear form $a(u,v) = c(\nabla u, \nabla v) - b(u,v)$.

b) (10 p)
a is clearly bilinear, and is continuous since

$$\begin{aligned} |a(u,v)| &\leqslant |c||u|_{H^1}|v|_{H^1} + |b|\|u\|_{L^2}\|v\|_{L^2} \\ &\leqslant \max(|c|,|d|)\left(|u|_{H^1} + \|u\|_{L^2}\right)\left(|v|_{H^1} + \|v\|_{L^2}\right) \\ &\leqslant 2\max(|c|,|d|)\|u\|_{H^1}\|v\|_{H^1}. \end{aligned}$$

For coercivity, we have

$$a(u,u) = c|u|_{H^1}^2 - b\|u\|_{L^2}^2.$$

Let C_Ω be the Poincaré constant for Ω (so that $\|u\|_{L^2} \leqslant C_\Omega |u|_{H^1(\Omega)}$ for all $u \in H^1_0(\Omega)$). If $b > 0$ then

$$a(u,u) \geqslant c|u|^2_{H^1} - bC^2_\Omega |u|^2_{H^1} = (c - bC^2_\Omega)|u|^2_{H^1},$$

so we need $c > bC^2_\Omega$. If $b \leqslant 0$ then

$$a(u,u) = c|u|^2_{H^1} + |b|\|u\|^2_{L^2} \geqslant c|u|^2_{H^1}$$

so we need $c > 0$. Thus, a is coercive provided

$$c > \max(0, bC^2_\Omega).$$

c) (10 p)
We end up with:

$$M\frac{\xi^{n+1} - \xi^n}{\Delta t} - bM\xi^{n+1} + cA\xi^{n+1} = 0$$

or

$$((1 - b\Delta t)M + c\Delta t A)\xi^{n+1} = M\xi^n$$

where $M_{ij} = (\varphi_i, \varphi_j)$ and $A_{ij} = (\nabla\varphi_i, \nabla\varphi_j)$.

Problem 3

a) (10 p)
Find $\xi \in \mathbb{R}^4$ such that $A\xi = b$, where

$$A = 5\begin{pmatrix} 2 & -1 & & \\ -1 & 2 & -1 & \\ & -1 & 2 & -1 \\ & & -1 & 2 \end{pmatrix}, \qquad b = \tilde{b} + 5e_4, \qquad \tilde{b}_i = \int_0^1 f(x)\varphi_i(x)\,dx$$

and $e_4 = (0,0,0,1)^T$.

b) (5+5=10 p)

(i) We have the basic error estimate

$$\|u - u_h\|_{H^1(\Omega)} \leqslant C\left(\sum_i h^2_{K_i}|u|^2_{H^2(K_i)}\right)^{1/2} = C\left(\sum_i h^2_{K_i}\|f\|^2_{L^2(K_i)}\right)^{1/2},$$

where $h_{K_i} \equiv 1/5$.

(ii) $|f|$ has a maximum at $x = 1$, so the error will be largest in the interval $K_5 = [0.8, 1]$, so we should put a node in this interval.

Problem 4 (10 p)
Cea's lemma.

Problem 5 (10 p)
Impose function values $v(x_i)$ and derivatives $v'(x_i)$ at the nodes x_i. Since each element domain K_i has two nodes, this constitutes 4 degrees of freedom, so we need at least $r = 3$.

Problem 6

a) (10 p)
It's easiest to find Ψ^{-1} first:

$$\Psi^{-1}(y) = (ay_1 + by_2,\ cy_2)$$

which gives

$$\Psi(x) = \left(\frac{x_1 - bx_2/c}{a},\ \frac{x_2}{c}\right).$$

We have

$$J = D\Psi^{-1} = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}, \qquad \det(J) = ac, \qquad J^{-1} = \frac{1}{ac}\begin{pmatrix} c & -b \\ 0 & a \end{pmatrix}.$$

b) (10 p)
Change of variables gives

$$\int_K \nabla\varphi_\alpha(x) \cdot \nabla\varphi_\beta(x)\, dx = \int_{\hat{K}} \nabla_y \hat{\varphi}_\alpha(y) \cdot J^{-1}J^{-T}\nabla_y \hat{\varphi}_\beta(y) |\det(J)|\, dy$$
$$= \frac{1}{ac}\int_{\hat{K}} \nabla_y \hat{\varphi}_\alpha(y) \cdot \begin{pmatrix} b^2+c^2 & -ab \\ -ab & a^2 \end{pmatrix} \nabla_y \hat{\varphi}_\beta(y)\, dy.$$

Problem 7

a) (10 p)
The Delaunay triangulation of $x_1, \ldots, x_N$ is a conforming triangulation $\mathcal{T}_h$ of $\mathrm{conv}(x_1, \ldots, x_N)$ (the convex hull of $x_1, \ldots, x_N$) such that the internal of the circumcircle of each triangle contains none of the points $x_1, \ldots, x_N$.

a) (10 p)