NTNU – Trondheim
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Science and Technology

Department of Mathematical Sciences

# Examination paper for
# **TMA4220 Numerical solution of partial differential equations using element methods**

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**Examination time (from–to):** 09:00–13:00

**Permitted examination support material:** C:
Calculator HP30S, CITIZEN SR-270X, CITIZEN SR-270X College, Casio fx-82ES PLUS.
K. Rottman: Matematisk formelsamling.
One yellow, stamped A4 sheet with own handwritten formulas and notes

**Language:** English

**Number of pages:** 4

**Number pages enclosed:** 0

**Checked by:**

Date Signature

**NB!** Justify your answers!

**Problem 1** Consider the two-dimensional Poisson problem

$$-\Delta u = f \quad \text{in} \quad \Omega,$$
$$u + \frac{\partial u}{\partial n} = g \quad \text{on} \quad \partial\Omega$$

and $f$ is a given function, and $\Omega$ is a bounded open set with a regular boundary.

**a)** The weak formulation for this problem is

$$\text{find } u \in V \text{ such that } a(u,v) = F(v), \quad \forall v \in V, \tag{1}$$

with $V = H^1(\Omega)$.

Find the expressions for $a$ and $F$.

Show that $a$ is positive, that is $a(v,v) > 0$ for all $v \neq 0$.

**b)** We now want to solve problem (1) using the Galerkin method on a finite dimensional subspace $V_h = \text{span}\{\varphi_1, \varphi_2, \cdots, \varphi_N\} \subset V$, that is

$$\text{Find } u_h \in V_h \text{ such that } a(u_h, v_h) = F(v_h), \quad \forall v_h \in V_h.$$

Show that this problem can be formulated as a linear equation system

$$A_h \mathbf{u} = \mathbf{b}_h$$

where $u_h = \sum_{i=1}^N u_i \varphi_i$ and $\mathbf{u} = [u_1, \cdots u_N]^T$.

Find expressions for $A_h$ and $\mathbf{b}_h$.

Prove that the stiffness matrix $A_h$ is symmetric positive definite.

**c)** Let $\Omega = (0,1) \times (0,1)$ be the unit square. Let $V_h = X_h^1$, the linear finite element space, defined on the grid given in Figure 1. Assume the gridsizes in both the vertical and horizontal directions to be $h$.

Find the stiffness element matrices $A^K$ of the the two elements $K_1$ and $K_2$ depicted in the figure.

**d)** Set up a set of sufficient conditions for the existence of a unique solution of a *general* problem of the form (1) (Lax-Milgram).

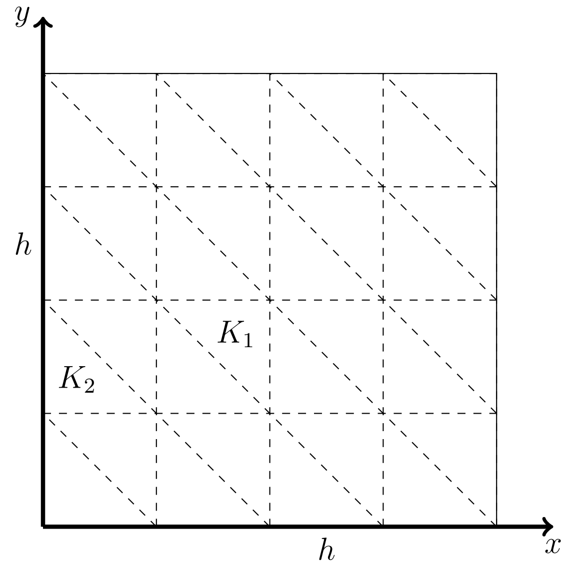


Figure 1: Triangulation of the unit square.

**e)** Let $\Pi_h^1 : C^{(0)}(\bar{\Omega}) \to X_h^1$ be the linear interpolation operator, defined on each element $K$ by $\Pi_h^1 v|_K = \Pi_K^1 v$, where

$$\Pi_K^1 v = v_1 \varphi_1|_K + v_2 \varphi_2|_K + v_3 \varphi_3|_K$$

where $v_i$ is the value of $v$ in each of the three vertices of the triangle $K$, and $\varphi_i|_K$ is the corresponding linear basis function restricted to $K$.

In the lectures, it was proved that

$$|v - \Pi_K^1 v|_{H^m(K)} \leq C_{K,m} \frac{h_K^2}{\rho_K^m} |v|_{H^2(K)}, \qquad m = 0, 1, \qquad \forall v \in H^2(\Omega)$$

where $h_K$ is the diameter and $\rho_K$ the sphericity of $K$. You can assume that $h_K/\rho_K \leq \delta$ for all $K$.

What are $\rho_K$, $h_K$ and $\delta$ for the grid of Figure 1? (If you do not remember how to calculate the sphericity, just indicate what it is by a figure).

Use the interpolation error bound above to prove an error bound of the form

$$\|u - u_h\|_{H^1(\Omega)} \leq C\,\bar{h}\,|u|_{H^2(\Omega)}$$

where $u_h \in X_h^1$ is the finite element solution, $\bar{h} = \max_K h_K$ and $C$ is a constant.

*Hint:* Use Céa lemma.

**Problem 2** According to the abstract definition, a finite element is characterized by three ingredients. Which ones?

**Problem 3** Integrals of a function $f$ over a triangular domain $K \in \mathbb{R}^2$ can be approximated by

$$\int_K f(\mathbf{z})d\Omega \approx |K| \sum_{q=1}^{N_q} \rho_q g(\mathbf{z}_q)$$

where $\rho_q$ are the weights and $\mathbf{z}_q$ are the vector quadrature points, and $|K|$ is the area of K.

A 3-point quadrature formula is given by

| $N_q$ | $\mathbf{z}_q$ | $\rho_q$ |
|---|---|---|
| 3 | $(1/2, 1/2, 0)$ | $1/3$ |
| | $(1/2, 0, 1/2)$ | $1/3$ |
| | $(0, 1/2, 1/2)$ | $1/3$ |

where the quadrature points $\mathbf{z}_q$ are given in barycentric coordinates.

Where on $K$ are the quadrature points located?

Use this quadrature formula to find an approximation to the integral

$$\int_K x \cdot \sin(\pi y/2) dxdy$$

where $K$ is the triangle with vertices $(0,0)$, $(1,0)$ and $(1/2,1)$.

**Problem 4** Given the one-dimensional eigenvalue problem:

$$-u_{xx} = \lambda u \qquad \text{in} \quad \Omega = (0,1), \qquad u(0) = u(1) = 0.$$

**a)** Set up the weak formulation of the problem, and use this to justify that all the eigenvalues are positive.

Verify that the values and corresponding functions

$$\lambda_j = \pi^2 j^2, \qquad u_j(x) = \sin(j\pi x), \qquad j = 1, 2 \ldots$$

satisfies the strong form of the eigenvalue problem.

**b)** Let the eigenvalue problem be solved by the linear finite element method, using $V_h = X_h^1$ with constant stepsize $h = 1/(N+1)$. This can be written as a generalized eigenvalue problem

$$M_h \mathbf{u} = \lambda_h A_h \mathbf{u}.$$

Find $M_h$ and $A_h$ in this case.

**c)** Find an expression for the eigenvalues $\lambda_{h,j}$ of the discrete problem.

Discuss how well the approximations $\lambda_{h,j}$ correspond to the exact values $\lambda_j$.

*Hints:*

- The matrices $M_h$ and $A_h$ have the same eigenvectors.
- The egienvalues of a symmetric, tridiagonal matrix of the form $C = \text{tridiag}\{b, a, b\} \in \mathbf{R}^{n\times n}$ are given by

$$\lambda_j(C) = a + 2b\cos\frac{j\pi}{n+1}, \qquad j = 1, 2, \cdots, n.$$