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Exam in TMA4220
Numerical Solution of Partial Differential Equations Using Element Methods

Friday December 9, 2011
Time: 15.00 – 19.00

Auxiliary materials: Simple calculator (Hewlett Packard HP30S or Citizen SR-270X)
All printed and hand written material.

Problem 1

Consider the Poisson equation

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega \in \mathbb{R}^2 \\ u &= g \quad \text{on } \Gamma_D \\ \frac{\partial u}{\partial n} &= \phi \quad \text{on } \Gamma_N \end{aligned} \tag{1}$$

where $\Gamma_N \cup \Gamma_D = \partial\Omega$ is the boundary of the domain Ω.

a) Establish the weak formulation:

$$\text{Find } \mathring{u} \in V \text{ such that } \quad a(\mathring{u}, v) = F(v) \quad \forall v \in V \tag{2}$$

corresponding to the problem above. In particular, identify the bilinear form a, the linear form F, the function space V, and explain how the solution u of (1) and the solution $\mathring{u}$ of (2) are related.

Comment: If you are unsure about how to deal with inhomogeneous boundary conditions, change them to something you are comfortable with, for example $u = 0$ on Γ_D, in which case $\mathring{u} = u$. But this simplification gives a slight reduction of the score of this point.

Solution:
Multiply the PDE with a test function v, integrate over the domain Ω and use Green's theorem. The result is

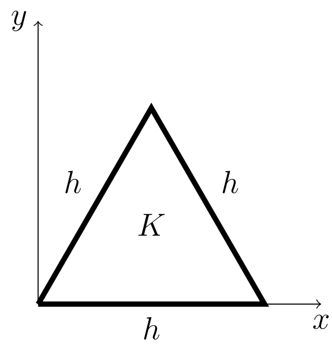
$$\int_\Omega \nabla u \cdot \nabla v \, d\Omega - \int_{\Gamma_N} \frac{\partial u}{\partial n} v \, ds - \int_{\Gamma_D} \frac{\partial u}{\partial n} v \, ds = \int_\Omega f v \, d\Omega.$$

Let $v \in V = H^1_{\Gamma_D}(\Omega) = \{v \in H^1(\Omega) \; : \; v = 0 \text{ on } \Gamma_D\}$. In this case, the integral over Γ_D is 0. Second, choose some $R_g \in H^1(\Omega)$, $R_g = g$ on Γ_D and $u = \mathring{u} + R_g$, where $\mathring{u} \in V$. In this case, the variational formulation is given by (2) with $V = H^1_{\Gamma_D}(\Omega)$,

$$a(\mathring{u}, v) = \int_\Omega \nabla \mathring{u} \cdot \nabla v \, d\Omega, \qquad F(v) = \int_\Omega f v \, d\Omega - \int_\Omega \nabla R_g \cdot \nabla v \, d\Omega + \int_{\Gamma_N} \phi v \, ds. \quad (3)$$

and finally $u = \mathring{u} + R_g$.

We would like to find an approximation to the solution of (2) by using a finite element method. As a reference element, we choose an equilateral triangle, with side length h. Use linear, nodal element functions, with one node in each corner of the element.



b) Find the three linear shape functions for the element K. Find the elemental stiffness matrix A^K and the elemental load vector $\mathbf{b}^K$.

Hint: The area of an equilateral triangle is $h^2\sqrt{3}/4$, and the volume of a pyramid is $BH/3$, where B is the area of the base and H the height.

Solution:
The three nodes are given by $N_0 = (0,0)$, $N_1 = (h,0)$ and $N_2 = (h/2, \sqrt{3}h/2)$. The corresponding three shape functions becomes:

$$\psi_0 = 1 - \frac{x}{h} - \frac{y}{\sqrt{3}h}, \qquad \psi_1 = \frac{x}{h} - \frac{y}{\sqrt{3}h}, \qquad \psi_2 = \frac{2y}{\sqrt{3}h}.$$

The gradients becomes

$$\nabla \psi_0 = \frac{1}{h}(-1, -\frac{1}{\sqrt{3}})^T, \qquad \nabla \psi_1 = \frac{1}{h}(1, -\frac{1}{\sqrt{3}})^T \qquad \nabla \psi_2 = \frac{1}{h}(0, \frac{2}{\sqrt{3}})^T$$

and the elements becomes

$$A^K_{00} = \int_K \nabla\psi_0 \cdot \nabla\psi_0 \, d\Omega = \frac{4}{3h^2}|K| = \frac{\sqrt{3}}{3}, \qquad A^K_{01} = \int_K \nabla\psi_0 \cdot \nabla\psi_1 \, d\Omega = -\frac{2}{3h^2}|K| = -\frac{\sqrt{3}}{6},$$

etc. Finally, the elemental matrix becomes

$$A^K = \frac{\sqrt{3}}{6}\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

Since f is still not known, we can only give a generic form of the elemental load vector $\mathbf{b}^K$, that is

$$b_j^K = \int_K f\psi_j \, d\Omega$$

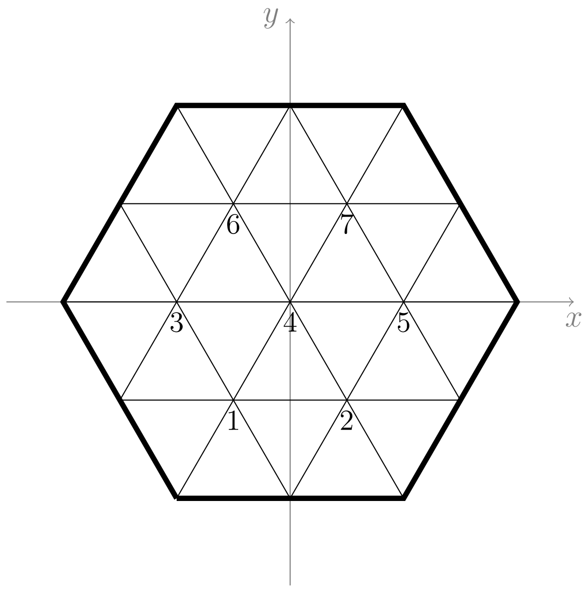
for $j = 0, 1, 2$. Boundary contributions are not included. If we anticipate the course of events (which you are not assumed to do on the exam) let $f = 1$. In this case, $b_j^K = \int_K \psi_j \, d\Omega$, which is exactly the volume of a pyramid with height 1 and a base K, which has the volume $h^2\sqrt{3}/4/3$, so

$$\textit{If } f = 1 \textit{ then} \quad \mathbf{b}^K = \frac{\sqrt{3}h^2}{12}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Now, consider the Poisson equation (1) with $f = 1$ and homogenous Dirichlet boundary conditions

$$u = 0 \quad \text{on} \quad \partial\Omega.$$

The domain Ω is a regular hexagon, and each edge is of length 1.
We will solve this problem by the finite element method, using the finite elements from **b)**, and with a triangulation based on equilateral triangles, see the figure.



c) Show that the finite element system

$$A_h\mathbf{u} = \mathbf{f}_h,$$

where A_h is the stiffness matrix, $\mathbf{u}$ the load vector, and $\mathbf{u}$ is a vector of the approximations of the numerical solutions in the nodes, can be written as

$$\begin{pmatrix} \alpha & \beta & \beta & \beta & 0 & 0 & 0 \\ \beta & \alpha & 0 & \beta & \beta & 0 & 0 \\ \beta & 0 & \alpha & \beta & 0 & \beta & 0 \\ \beta & \beta & \beta & \alpha & \beta & \beta & \beta \\ 0 & \beta & 0 & \beta & \alpha & 0 & \beta \\ 0 & 0 & \beta & \beta & 0 & \alpha & \beta \\ 0 & 0 & 0 & \beta & \beta & \beta & \alpha \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \end{pmatrix} = \begin{pmatrix} \gamma \\ \gamma \\ \gamma \\ \gamma \\ \gamma \\ \gamma \\ \gamma \end{pmatrix}$$

Find α, β and γ.

Solution:
We know that $A_{h,i,j} = \sum_K A^K_{\cdot,\cdot}$, *that is,* $A_{h,ij}$ *is the sum of the corresponding elements for the elemental matrices from elements having* i, j *as nodes. Similar for* $\mathbf{f}_{h,i}$. *Thus*

$$\alpha = 6 \cdot \frac{\sqrt{3}}{3} = 2\sqrt{3}, \qquad \beta = -2 \cdot \frac{\sqrt{3}}{6} = -\frac{\sqrt{3}}{3}, \qquad \gamma = 6 \cdot \frac{\sqrt{3}h^2}{12} = \frac{\sqrt{3}h^2}{2}.$$

with $h = 0.5$.

d) Let us change the boundary conditions of (1) to

$$u = -x \quad \text{on} \quad \partial\Omega.$$

How will you solve this problem using the finite element method above? Demonstrate the idea by rewriting the first equation (row) of the finite element system to include contributions from the nonhomogeneous boundary conditions.
Solution:
The discrete solution can be written as

$$u_h = \mathring{u}_h + R_{g,h} = \sum_{j \in \mathcal{N}} u_j \varphi_j + \sum_{j \in \partial\mathcal{N}} u_j \varphi_j$$

where $\mathcal{N}$ *represents the nodes inside the domain* Ω *and* $\partial\mathcal{N}$ *those on the boundary, in the latter the values of* u_i *are known. Using the standard procedure, the discrete variational form becomes*

$$\sum_{j \in \mathcal{N}} \left(\int_\Omega \nabla\varphi_i \nabla\varphi_j \, d\Omega \right) u_j = \int f \varphi_i \, d\Omega - \sum_{j \in \partial\mathcal{N}} \left(\int_\Omega \nabla\varphi_i \nabla\varphi_j \, d\Omega \right) u_j, \qquad i \in \mathcal{N}.$$

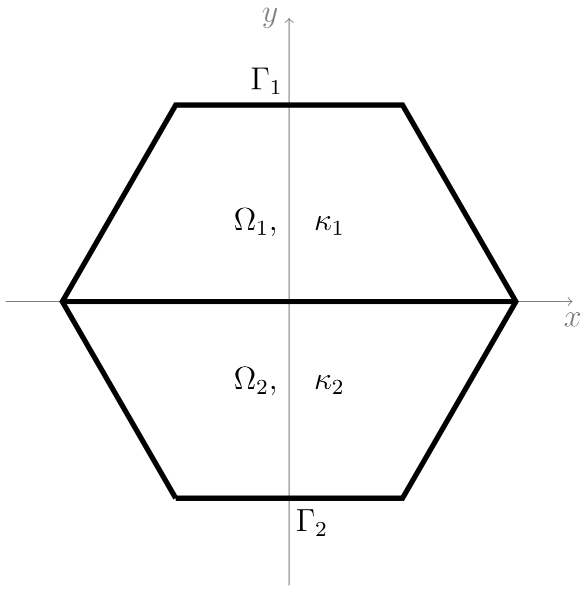
So we got the contributions from the boundary conditions on the right hand side. For $i = 1$, *we get contributions from the nodes* $(0, -\sqrt{3}/2)$, $(-1/2, -\sqrt{3}/3)$ *and* $(-3/4, -\sqrt{3}/4)$. *So, the first element in the right hand side becomes:*

$$f_{h,1} = \gamma - \beta \cdot 0 - \beta \cdot \frac{1}{2} - \beta \cdot \frac{3}{4} = \frac{13}{24}\sqrt{3}.$$

Consider the Poisson problem again, but this time with domain composed by two subdomains, separated by the x-axis, and with different diffusion constants. More specific, consider

$$\begin{aligned} -\kappa_1 \, \Delta u &= f && \text{in} \quad \Omega_1 \\ -\kappa_2 \, \Delta u &= f && \text{in} \quad \Omega_2 \\ u &= 0 && \text{on} \quad \Gamma_1 \\ \frac{\partial u}{\partial n} &= \phi && \text{on} \quad \Gamma_2 \\ \kappa_1 \frac{\partial u}{\partial y} &= \kappa_2 \frac{\partial u}{\partial y} && \text{for} \quad y = 0 \end{aligned}$$

with $\kappa_1, \kappa_2 > 0$.



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e) Establish the weak formulation for this problem, and discuss existence and uniqueness of the solution.

Can the weak formulation be reformulated as a minimization problem? If yes, which one? Give a reason for your answer.

Solution:

As usual, multiply with a test function v, integrate over the domain Ω and use Greens theorem on each subdomain“”

$$-\kappa_1 \int_{\Omega_1} \Delta u v \, d\Omega - \kappa_2 \int_{\Omega_2} \Delta u v \, d\Omega = \int_{\Omega} f v \, d\Omega$$

or

$$\begin{aligned} &\kappa_1 \left(\int_{\Omega_1} \nabla u \cdot \nabla v d\Omega - \int_{\Gamma_1} \frac{\partial u}{\partial n_1} v \, ds - \int_{\Gamma_0} \frac{\partial u}{\partial n_1} v \, ds \right) \\ +&\kappa_2 \left(\int_{\Omega_2} \nabla u \cdot \nabla v d\Omega - \int_{\Gamma_2} \frac{\partial u}{\partial n_2} v \, ds - \int_{\Gamma_0} \frac{\partial u}{\partial n_2} v \, ds \right) = \int_{\Omega} f v \, d\Omega. \end{aligned}$$

where Γ_0 is the the intersection of Ω and the x-axis. Since $n_1 = -n_2$ the boundary condition on $y = 0$ makes the two integrals over Γ_0 to cancel out. By requiring $v = 0$ on Γ_1 the integral over this part of the boundary becomes 0. The variational form becomes

$$\text{Find } u \in H^1_{\Gamma_1}(\Omega) \text{ such that} \qquad a(u, v) = F(v) \qquad \forall v \in H^1_{\Gamma_1}(\Omega) \tag{4}$$

where

$$\begin{aligned} a(u, v) &= \kappa_1 \int_{\Omega_1} \nabla u \cdot \nabla v \, d\Omega + \kappa_2 \int_{\Omega_2} \nabla u \cdot \nabla v \, d\Omega \\ F(v) &= \int_{\Omega} f v \, d\Omega + \kappa_2 \int_{\Gamma_2} \phi v \, ds, \\ H^1_{\Gamma_1}(\Omega) &= \{ v \in H^1(\Omega) \, : \, v = 0 \text{ on } \Gamma_1 \}. \end{aligned}$$

According to Lax-Milgram's lemma, (4) has a unique solution if

- *$H^1_{\Gamma_1}(\Omega)$ is a Hilbert space (OK).*
- *a is bilinear (OK), continuous (OK) and coercive (se below).*
- *F is linear (OK) and continuous (OK).*

But we need to prove that a is coercive, that is there exist an $\alpha > 0$ such that

$$a(v, v) \geq \alpha \|v\|^2_{H^1(\Omega)}, \qquad \forall v \in H^1_{\Gamma_1}(\Omega) \tag{5}$$

Let $\kappa = \min\{\kappa_1, \kappa_2\} > 0$. Then, for all $v \in H^1_{\Gamma_1}(\Omega)$:

$$a(v, v) \geq \kappa \int_{\Omega} \nabla v \cdot \nabla v \, d\Omega = \kappa |v|^2_{H^1(\Omega)}$$

From Poincaré inequality we know there exist a constant $C_\Omega > 0$ such that

$$\|v\|^2_{L^2(\Omega)} \leq C_\Omega |v|^2_{H^1(\Omega)}, \qquad \forall v \in H^1_{\Gamma_1}(\Omega)$$

and then

$$\|v\|^2_{H^1(\Omega)} = \|v\|^2_{L^2(\Omega)} + |v|^2_{H^1(\Omega)} \leq (1 + C_\Omega) |v|^2_{H^1(\Omega)}$$

So (5) *is satisfied, with* $\alpha = \kappa/(1 + C_\Omega)$.

Finally $a(u, v) = a(v, u)$ *so there is an equivalent minimization problem:*

$$\textit{Find } u \in H^1_{\Gamma_1}(\Omega) \textit{ such that} \qquad J(u) = \min J(v), \qquad \forall v \in H^1_{\Gamma_1}(\Omega)$$

with

$$J(v) = \frac{1}{2}a(v, v) - F(v).$$

Problem 2

Consider the problem

$$0.01\, u_{xx} + 2\, u_x = 0, \qquad u(0) = 0, \quad u(1) = 1.$$

This problem is solved by a linear finite element method with nodal basis functions on a uniform grid, that is $V_h = X^1_h$ with constant $h = 1/M$.

a) Show that the finite element method can be expressed as

$$A_h\mathbf{u} + C_h\mathbf{u} = \mathbf{f}$$

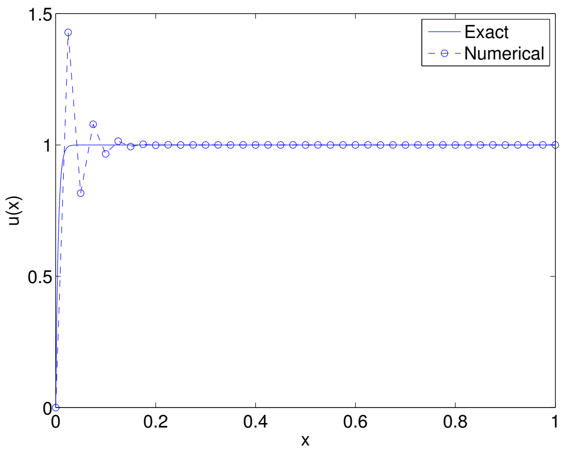
and find the matrices A_h, C_h and the vector $\mathbf{f}$. Here, A_h represents the diffusion term and C_h the advection term.

Solution:

Similar problems has been solved several times, see also sec. 11.2 in the book. We get

$$\begin{aligned} A_h &= \frac{1}{100h}\,\textit{tridiag}\{1, -2, 1\} \in \mathbb{R}^{(M-1)\times(M-1)} \\ C_h &= \textit{tridiag}\{-1, 0, 1\} \in \mathbb{R}^{(M-1)\times(M-1)} \\ \mathbf{f} &= (0, \cdots, 0, 1/(100h) - 1)^T \in \mathbb{R}^{M-1} \end{aligned}$$

The exact and the numerical solution, with $M = 40$ elements is given in the figure below.



b) Explain why the numerical solution oscillates.

How many (equal) elements are needed to avoid oscillations?

Solution:

The linear system above can be written as a difference equation:

$$(\frac{1}{100h}+1)u_{i+1}-\frac{2}{h}u_i+(\frac{1}{100h}-1)u_{i-1}, \qquad i=1,2,\ldots,M-1$$

with $u_0 = 0$ *and* $u_M = 1$. *We search for solutions on the form* $u_i = \rho^i$, *there are two of them:*

$$\rho_1 = 1, \qquad \rho_2 = \frac{1-100h}{1+100h}$$

and the exact solution of the difference scheme can be written as

$$u_i = a_1\rho_1^i + a_2\rho_2^i$$

where a_1 *and* a_2 *can be found from the boundary conditions. However, if* $h > 1/100$ *then* ρ_2 *becomes negative, and the solution starts to oscillates. To avoid the problem, we have to require* $h \leq 1/100$, *or* $M \geq 100$.

c) Explain how we can improve the numerical solution based on linear elements on a uniform grid, without increasing the number of elements.

Solution:

The idea is to add artifical diffusion, so that the oscillation no longer occur, that is solving the problem

$$(0.001+\phi)u_{xx} + 2u_x = 0$$

Repeating the analysis (or consider the local Péclet numer directly), tell us that we will have to require $h < (1/100+\phi)$ *which is true for all* h *if for instance* $\phi = h$. *Which corresponds to a downwind scheme.*

Problem 3

*Point **a)** and **b)** are given half weight.*

Consider the one-dimensional eigenvalue problem

$$\beta u_x = \lambda u \quad \text{in} \quad \Omega = (0,1), \qquad u(0) = u(1), \tag{6}$$

where β is a given constant. Notice the periodic boundary conditions.

a) Verify that the eigenfunctions u_n and corresponding eigenvalues λ_n,

$$u_n(x) = e^{ik_nx}, \qquad \lambda_n = i\beta k_n, \qquad \text{with } k_n = 2\pi n, \quad i=\sqrt{-1}$$

satisfy (6) for $n = 0, \pm1, \pm2, \ldots$

Solution:

That $u_n = e^{ik_nx}$ *solves the equation* $\beta u_x = \lambda u$ *is proved by insertion. To satisfy the boundary conditions,* k_n *have to be some mutiple of* 2π.

It can be proved that the weak formulation of the eigenvalue problem (6) is given by:

$$\text{Find } u \in V \text{ and } \lambda \in \mathbb{C} \quad \text{such that} \quad c(u,v) = \lambda(u,v), \quad \forall v \in V$$

with

$$V = \{v \in H^1(\Omega) \mid v(0) = v(1)\}, \quad c(w,v) = \int_0^1 \beta w_x v \, dx, \quad (w,v) = \int_0^1 wv \, dx.$$

b) Show that the bilinear form c is skew-symmetric, that is

$$c(w,v) = -c(v,w).$$

Solution:

$$\begin{aligned} c(w,v) &= \beta \int_0^1 w_x v \, dx = -\beta \int_0^1 w v_x \, dx + v(1)u(1) - v(0)u(0) \\ &= -\beta \int_0^1 w v_x \, dx = -c(v,w) \end{aligned}$$

Following the standard Galerkin method, the discrete eigenvalue problem can be written as

$$\text{Find } u_h \in V_h \subset V \text{ and } \lambda_h \in \mathbb{C} \quad \text{such that} \quad c(u_h,v) = \lambda(u_h,v), \quad \forall v \in V_h. \qquad (7)$$

c) Let $V_h = X_h^1$ with constant $h = 1/M$. Show that (7) can be written as a linear system of algebraic equations

$$C_h \mathbf{u} = \lambda_h M_h \mathbf{u},$$

and find the matrices C_h and M_h.
In particular, explain how to handle the periodic boundary conditions.
Solution:
Let φ_i, $i = 0, 1, \cdots, M$ be the nodal basis functions of X_h^1. To account for the periodic boundary conditions, extend φ_0 and φ_M to one step outside the interval (0,1), and notice that $\varphi_0(x) = \varphi_M(1+x)$. Since $u_0 = u_M$ the numerical solution can be written as

$$u_h(x) = \sum_{j=1}^{M} u_j \varphi_j(x) = \sum_{j=0}^{M-1} u_j \varphi_j(x).$$

The problem (7) *then becomes*

$$\sum_{j=1}^{M} c(\varphi_j, \varphi_i) u_j = \lambda_h \sum_{j=1}^{M} (\varphi_j, \varphi_i), i = 1, 2, \cdots, M$$

$$C_{h,ij} = c(\varphi_j, \varphi_i) = \beta \int_0^1 \frac{\partial \varphi_j}{\partial x} \varphi_i \, dx = \begin{cases} -\beta/2 & j = i-1 \\ +\beta/2 & j = i+1 \\ 0 & \textit{otherwise} \end{cases}$$

$$M_{h,ij} = (\varphi_j, \varphi_i) u_j = \begin{cases} h/6 & j = i \pm 1 \\ 4h/6 & j = i \\ 0 & \textit{otherwise} \end{cases}$$

for $i, j = 1, 2, 3, \cdots, M-1$. *In addition* $C_{h,1M} = -C_{h,M1} = -\beta/2$ *and* $M_{h,1M} = M_{h,M1} = h/6$. *So the matrices are no longer tridiagonal.*