TMA4220: Numerical solution of partial differential equations by element methods

Spectra of the continuous and discrete Laplace operator

March 19, 2002

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1 Strong form

Consider the one-dimensional eigenvalue problem

$$-u_{xx} = \lambda u \quad in \ \ \Omega = (0,1) \ , \quad u(0) = u(1) = 0 \ . \tag{1}$$

The eigenfunctions

$$u_j(x) \quad = \quad sin(\pi\, j\, x) \tag{2}$$

and eigenvalues

$$\lambda_j \quad = \quad \pi^2 j^2 \tag{3}$$

satisfy the eigenvalue problem (1) for $j = 1, 2, ..., \infty$.

2 Weak form

The weak form of (1) is: Find $u \in X = H_0^1(\Omega)$ and $\lambda \in I\!R$ such that

$$\int_0^1 u_x v_x \, dx = \lambda \int_0^1 u\, v\, dx \quad \forall v \in X \ . \tag{4}$$

We can also express this weak form as: Find $(u, \lambda) \in X \times I\!R$ such that

$$a(u,v) = \lambda(u,v) \quad \forall v \in X \tag{5}$$

where

$$a(w,v) \quad = \quad \int_0^1 w_x v_x \, dx \tag{6}$$

$$(w,v) \quad = \quad \int_0^1 w\, v\, dx \tag{7}$$

Note that $a(w,v)$ and (w,v) are both symmetric, positive-definite bilinear forms.

3 Rayleigh quotient

From the weak formulation of the eigenvalue problem, it follows that

$$\forall v \in X = H_0^1(\Omega)\, , \qquad \lambda = \frac{a(v,v)}{(v,v)} > 0 \tag{8}$$

since $a(\cdot,\cdot)$ and $(\cdot,\cdot)$ are SPD bilinear forms. In particular,

$$\lambda_j = \frac{a(u_j,u_j)}{(u_j,u_j)} > 0\, , \ \ j = 1, 2, ..., \infty\, . \tag{9}$$

Note that

$$\frac{a(v,v)}{(v,v)} \geq \lambda_{\min} > 0 \ , \tag{10}$$

with

$$\lambda_{\min} = \min_{v \in X} \frac{a(v,v)}{(v,v)} \ . \tag{11}$$

4 Discrete eigenvalue problem

Following the standard Galerkin procedure, we can express the discrete eigenvalue problem as: Find $u_h \in X_h \subset X = H_0^1(\Omega)$ and $\lambda_h \in I\!R$ such that

$$a(u_h, v) = \lambda_h (u_h, v) \quad \forall v \in X_h \ . \tag{12}$$

The discrete eigenfunctions and eigenvalues are denoted as $(u_h)_j$ and $(\lambda_h)_j$, respectively. If $\dim(X_h) = N$, we immediately observe that

$$(\lambda_h)_j = \frac{a((u_h)_j, (u_h)_j)}{((u_h)_j, (u_h)_j)} > 0 \, , \ j = 1, 2, ..., N \, . \tag{13}$$

In particular,

$$\forall v \in X_h \, , \qquad 0 < (\lambda_h)_{\min} \leq \frac{a(v,v)}{(v,v)} \leq (\lambda_h)_{\max} \ . \tag{14}$$

Since

$$(\lambda_h)_{\min} = \min_{v \in X_h} \frac{a(v,v)}{(v,v)} \ , \tag{15}$$

while

$$\lambda_{\min} = \min_{v \in X} \frac{a(v,v)}{(v,v)} \ , \tag{16}$$

we obtain the important result that

$$(\lambda_h)_{\min} \geq \lambda_{\min} \ . \tag{17}$$

In other words, the minimum eigenvalue for the discrete problem is always greater than or equal to the minimum eigenvalue for the continuous problem.

5 One-dimensional example

We consider here the numerical solution of (1) using linear finite elements. In particular, we assume that we use K equal elements, each of length (mesh size) $h = 1/K$. Our discrete space X_h can then be defined as

$$
\begin{aligned}
X_h &= \{v \in X = H_0^1(\Omega) \mid v_{|_{T_h^k}} \in \mathbb{P}_1(T_h^k),\ k = 1, ..., K\} && (18)\\
&= span\{\phi_1, \phi_2, ..., \phi_N\} \ . && (19)
\end{aligned}
$$

As usual, we assume that we use a nodal basis for X_h, that is,

$$\forall v \in X_h\,, \qquad v(x) = \sum_{i=1}^{N} v_i\, \phi_i(x), \tag{20}$$

where the basis coefficients $v_i = v(x_i)\,,\ i = 1, 2, .., N$. Note that $K = N + 1$, implying that $h = 1/(N + 1)$. Also note that $v(x_0) = v(x_{N+1}) = 0$ due to the homogenous Dirichlet boundary conditions.

Using the nodal basis, the discrete eigenvalue problem (12) can be expressed as the following system of algebraic equations:

$$\underline{A}_h\, \underline{u}_h = \lambda_h\, \underline{M}_h\, \underline{u}_h \ , \tag{21}$$

where

$$u_h(x) = \sum_{i=1}^{N} u_{hi}\, \phi_i(x) \tag{22}$$

and

$$\underline{u}_h = [u_{h1}, u_{h2}, ..., u_{hN}]^T \ . \tag{23}$$

Since the matrix elements are given as

$$
\begin{aligned}
(A_h)_{mn} &= a(\phi_m, \phi_n) \ , && (24)\\
(M_h)_{mn} &= (\phi_m, \phi_n) \quad , \ 1 \le m, n \le N && (25)
\end{aligned}
$$

the stiffness matrix $\underline{A}_h$ can be written as the tridiagonal SPD matrix (e.g., in the case where $N = 5$)

$$\underline{A}_h = \frac{1}{h} \begin{pmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 2 \end{pmatrix}, \tag{26}$$

while the mass matrix $\underline{M}_h$ is given as

$$\underline{M}_h = \frac{h}{6} \begin{pmatrix} 4 & 1 & 0 & 0 & 0 \\ 1 & 4 & 1 & 0 & 0 \\ 0 & 1 & 4 & 1 & 0 \\ 0 & 0 & 1 & 4 & 1 \\ 0 & 0 & 0 & 1 & 4 \end{pmatrix} . \tag{27}$$

The algebraic eigenvalue problem (21) has N eigenvalues $(\lambda_h)_j$, $j = 1, ..., N$, with corresponding eigenvectors $(\underline{u}_h)_j$, $j = 1, ..., N$. In this particular one-dimensional case, the eigenvectors are the same as the continuous eigenfunctions $u_j(x)$ evaluated at the nodal points $x_i = ih$

$$(u_{hi})_j = u_j(x_i) = sin(\pi\, j\, (ih)) \quad i = 1, .., N, \;\; j = 1, ..., N\,. \tag{28}$$

Note that, in general, this results is *not* true. Also note that the eigenvectors are compatible with the homogeneous boundary conditions ($i = 0$ and $i = N + 1$).

Operating with $\underline{A}_h$ on the eigenvector $(\underline{u}_h)_j$, $j = 1, ..., N$, gives [1]

$$\frac{1}{h}[-sin(\pi\, j\, (i-1)h) + 2\, sin(\pi\, j\, ih) - sin(\pi\, j\, (i+1)h)] =$$
$$sin(\pi\, j\, ih) \cdot \frac{2}{h}[1 - cos(\pi\, j\, h)] \;\; , i = 1, ..., N. \tag{29}$$

Hence,

$$\lambda_j(\underline{A}_h) \equiv (\lambda_h)_j = \frac{2}{h}[1 - cos(\pi\, j\, h)] \;\; , j = 1, ..., N. \tag{30}$$

Note that $\lambda_j(\underline{A}_h)$ here denotes the eigenvalue corresponding to the eigenvalue problem

$$\underline{A}_h\, (\underline{u}_h)_j = (\lambda_h)_j\, (\underline{u}_h)_j \;\; , \tag{31}$$

and *not* the generalized eigenvalue problem (21).

By a similar procedure, we can also show that

$$\lambda_j(\underline{M}_h) = \frac{h}{3}[2 + cos(\pi\, j\, h)] \;\; , j = 1, ..., N. \tag{32}$$

where $\lambda_j(\underline{M}_h)$ denotes the eigenvalue corresponding to the eigenvalue problem

$$\underline{M}_h\, (\underline{u}_h)_j = (\lambda_h)_j\, (\underline{u}_h)_j \;\; . \tag{33}$$

We note that, in this *particular* case, $(\underline{u}_h)_j$, $j = 1, ..., N$ are eigenvectors of *both* $\underline{A}_h$ and $\underline{M}_h$; in general, this will *not* be true.

[1] Recall that $sin\,(\alpha - \beta) + sin\,(\alpha + \beta) = 2\, sin\, \alpha\, cos\, \beta$

For the lowest eigenmodes, i.e., $jh \ll 1$ (or $j/N \ll 1$), we obtain

$$\lambda_j(\underline{A}_h) = \frac{2}{h}[1 - (1 - \pi^2 j^2 h^2/2 + ...)] \approx \pi^2 j^2 \, h \ , \tag{34}$$

while

$$\lambda_j(\underline{M}_h) \approx h \ . \tag{35}$$

For large values of jh, $jh \sim \mathcal{O}(1)$, we obtain

$$\lambda_j(\underline{A}_h) \approx 4/h \ , \tag{36}$$

$$\lambda_j(\underline{M}_h) \approx h/3 \ . \tag{37}$$

It thus follows that the condition number of $\underline{A}_h$ is

$$\kappa(\underline{A}_h) = \frac{\lambda_{\max}(\underline{A}_h)}{\lambda_{\min}(\underline{A}_h)} \approx \frac{4/h}{\pi^2 h} = \frac{4}{\pi^2} \, h^{-2} \ \sim \mathcal{O}(h^{-2}) \tag{38}$$

as advertised earlier.

Similarly, we obtain that the condition number of $\underline{M}_h$ is

$$\kappa(\underline{M}_h) = \frac{\lambda_{\max}(\underline{M}_h)}{\lambda_{\min}(\underline{M}_h)} \approx \frac{h}{h/3} = 3 \ \sim \mathcal{O}(1) \ . \tag{39}$$

Finally, because $\underline{A}_h$ and $\underline{M}_h$ have the *same* set of eigenvectors $(\underline{u}_h)_j$, $j = 1, ..., N$ in this particular one-dimensional case, we can easily find the eigenvalues of the generalized eigenvalue problem (21) as

$$(\lambda_h)_j = \lambda_j(\underline{M}_h^{-1}\underline{A}_h) = \frac{\lambda_j(\underline{A}_h)}{\lambda_j(\underline{M}_h)} = \frac{6}{h^2} \cdot \frac{(1 - cos(\pi \, j \, h))}{(2 + cos(\pi \, j \, h))} \ , j = 1, ..., N. \tag{40}$$

For the lowest eigenmodes, $jh \ll 1$, we obtain

$$\lambda_j(\underline{M}_h^{-1}\underline{A}_h) \approx \pi^2 \, j^2 \ , \tag{41}$$

while for the highest eigenmodes, $jh \sim \mathcal{O}(1)$, we obtain

$$\lambda_j(\underline{M}_h^{-1}\underline{A}_h) \approx 12/h^2 \approx 12 \, N^2 \ . \tag{42}$$

Since the eigenvalues for the continuous case (1) are

$$\lambda_j = \pi^2 \, j^2 \ , \ j = 1, 2, ..., \infty \, , \tag{43}$$

the discrete eigenvalues $\lambda_j(\underline{M}_h^{-1}\underline{A}_h)$ agree very well for the low eigenmodes. However, the highest discrete eigenvalue is approximately $12 \, N^2$, while the corresponding continuous eigenvalue is $\pi^2 \, N^2 \approx 10 \, N^2$.

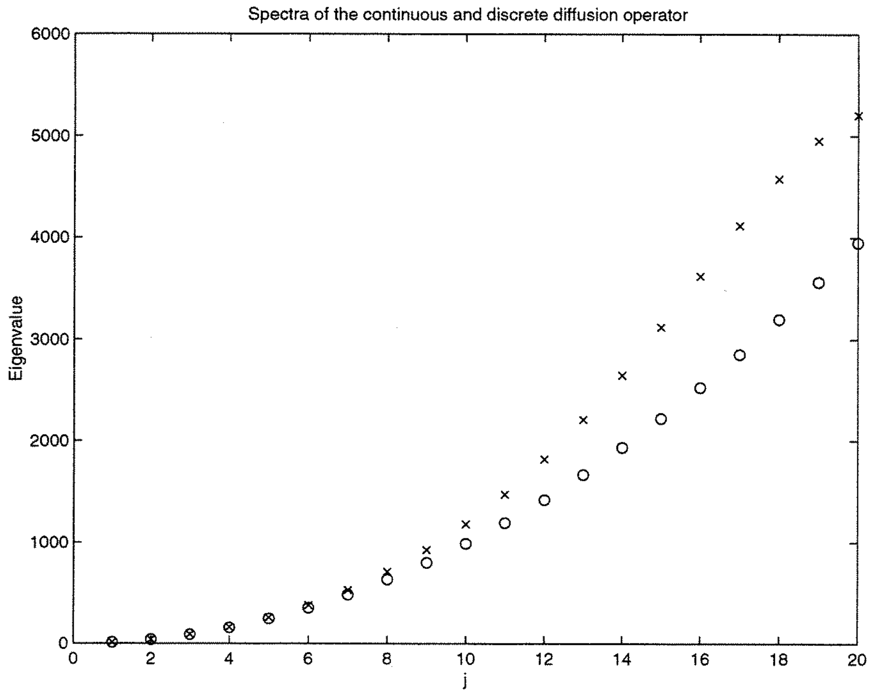

Figure 1: A comparison between the discrete eigenvalues (40) ($\times$) and the continuous eigenvalues (3) ($\circ$) in the case when $N = 20$. Note that only the first N eigenvalues are plotted for the continuous case.

6 Remarks

1. Note that the result $\kappa(\underline{A}_h) \sim \mathcal{O}(h^{-2})$ is the relevant result when we want to estimate the number of iterations required in order to solve the system $\underline{A}_h \underline{u}_h = \underline{F}_h$ using the conjugate gradient method ($\mathcal{N}_{iter} \sim \mathcal{O}(\sqrt{\kappa(\underline{A}_h)}) \sim \mathcal{O}(h^{-1})$).

2. The results $\kappa(\underline{A}_h) \sim \mathcal{O}(h^{-2})$ and $\kappa(\underline{M}_h) \sim \mathcal{O}(1)$ extend to the multi-dimensional case for linear elements (assuming the same discretization in each spatial direction).

3. The result for $\lambda_{max}(\underline{M}_h^{-1}\underline{A}_h)$ plays an important role when solving the unsteady heat equation using explicit time integration in combination with a finite element discretization is space. We will return to this case later.

7 Connection to physical applications

Consider a system consisting of a mass m attached to a spring of stiffness k. Assume that the point mass is in a position $y = 0$ at equilibrium; see Figure 2.

The governing equation for free, undamped vibrations about the equilibrium position follows from Newton's law of motion

$$m\,\ddot{y} + k\,y = 0 \ . \tag{44}$$

The solution will be of the form

$$y(t) \sim y_0\, e^{i\omega_0 t} \tag{45}$$

with

$$\omega_0 = \sqrt{\frac{k}{m}} \ . \tag{46}$$

For a more complex system involving multiple masses and springs, the governing equation can be expressed as a system

$$\underline{M}\,\ddot{\underline{y}} + \underline{A}\,\underline{y} = \underline{0} \ , \tag{47}$$

where $\underline{M}$ is a matrix dependent upon the individual masses, and $\underline{A}$ is a matrix dependent upon the spring stiffnesses. Again, we can assume that the solution will be of the form

$$\underline{y}(t) \sim \underline{y}_0\, e^{i\omega_0 t} \ , \tag{48}$$

implying that

$$-\omega_0^2\,\underline{M}\,\underline{y}_0 + \underline{A}\,\underline{y}_0 = \underline{0} \ , \tag{49}$$

or

$$\underline{A}\,\underline{y}_0 = \omega_0^2\,\underline{M}\,\underline{y}_0 \ . \tag{50}$$

This is an analogue to our discrete eigenvalue problem, with the eigenvalue representing the square of the eigenfrequency.

Note that, for our *discrete* approximation (12) to the continuous eigenvalue problem (1), we only expect the *lowest* eigenmodes (or eigenfunctions) and eigenvalues to be good approximations to the corresponding physical quantities. This is, indeed, the case.

Figure 2

Figure missing.

8 Exercises

1. For our one-dimensional sample problem, we found that $\underline{A}_h$ and $\underline{M}_h$ have the same set of eigenvectors. Show that, in this particular case, $\lambda_j(\underline{M}_h^{-1}\underline{A}_h) = \frac{\lambda_j(\underline{A}_h)}{\lambda_j(\underline{M}_h)}$, i.e., derive the results in (40).

2. For the lowest eigenmodes ($jh \ll 1$), show that the discrete eigenvalues given in (40) have an error which is $\mathcal{O}(h^2)$ compared to the corresponding eigenvalues for the continuous case. Is the minimum discrete eigenvalue larger or smaller than the minimum eigenvalue for the continuous problem?

3. Let $\underline{M}_h \in I\!R^{N\times N}$ be the one-dimensional finite element mass matrix using linear elements. Consider solving the system $\underline{M}_h\,\underline{z} = \underline{r}$ using the conjugate gradient method. Estimate the number of iterations.

4. Prove (14) in the case when $\dim(X_h) = N$. Hint: Consider expanding $v \in X_h$ in terms of the eigenfunctions $(u_h)_j$, $j = 1, ..., N$.

5. Assume that the continuous eigenvalue problem (1) is discretized using a standard finite difference scheme on a uniform mesh with N internal nodes. What are the discrete eigenvalues in this case? Is the minimum discrete eigenvalue larger or smaller than the minimum eigenvalue for the continuous problem?

6. Prove (29) and (30).