10 Deformed geometries

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MA8502 Numerical solution of partial differential equations

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1 The single-domain case

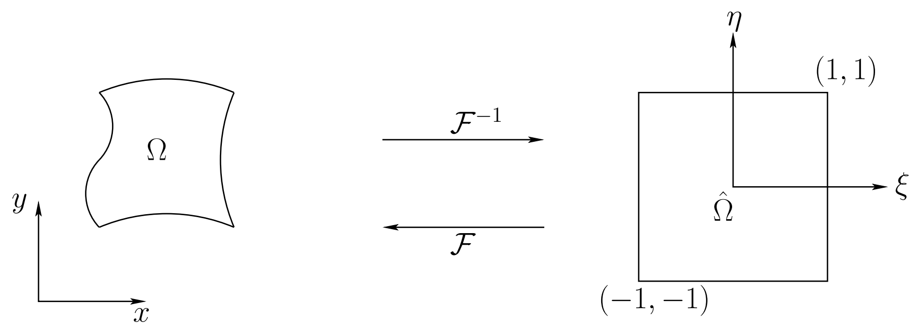


Figure 1: The two-dimensional deformed domain Ω is here considered to be the image of the reference domain $\hat{\Omega}$ through a regular one-to-one mapping, $\mathcal{F}$.

Consider the two-dimensional domain Ω depicted in Figure 1. A point $(x, y) \in \Omega$ can be considered as coming from one and only one point $(\xi, \eta) \in \hat{\Omega}$. In general, we have

$$x = x(\xi, \eta), \tag{1}$$
$$y = y(\xi, \eta). \tag{2}$$

Conversly, we have

$$\xi = \xi(x, y), \tag{3}$$
$$\eta = \eta(x, y). \tag{4}$$

The total differentials $\mathrm{d}x$ and $\mathrm{d}y$ can thus be expressed as

$$\mathrm{d}x = \frac{\partial x}{\partial \xi}\,\mathrm{d}\xi + \frac{\partial x}{\partial \eta}\,\mathrm{d}\eta,$$
$$\mathrm{d}y = \frac{\partial y}{\partial \xi}\,\mathrm{d}\xi + \frac{\partial y}{\partial \eta}\,\mathrm{d}\eta.$$

We can also express these differentials in matrix form as

$$\begin{pmatrix} \mathrm{d}x \\ \mathrm{d}y \end{pmatrix} = \underbrace{\begin{pmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} \end{pmatrix}}_{\underline{J}=\text{Jacobian}} \begin{pmatrix} \mathrm{d}\xi \\ \mathrm{d}\eta \end{pmatrix} = \underline{J} \begin{pmatrix} \mathrm{d}\xi \\ \mathrm{d}\eta \end{pmatrix}, \tag{5}$$

where we have introduced the Jacobian (a two-by-two matrix) associated with the mapping $\mathcal{F}$. Note that the Jacobian will, in general, be different at every point in space.

From the Jacobian matrix introduced earlier, we can compute the determinant

$$J \equiv \det(\underline{J}) = \frac{\partial x}{\partial \xi}\frac{\partial y}{\partial \eta} - \frac{\partial x}{\partial \eta}\frac{\partial y}{\partial \xi}. \tag{6}$$

The determinant of $\underline{J}$ represents the ratio of an area element $\mathrm{d}\Omega$ in Ω to the corresponding area element $\mathrm{d}\hat{\Omega}$ in $\hat{\Omega}$, i.e.,

$$J \equiv \det(\underline{J}) = \frac{\mathrm{d}\Omega}{\mathrm{d}\hat{\Omega}}.$$

In order to see this, consider Figure 2. The two vectors v_1 and v_2 aligned with two of the edges of surface element $\mathrm{d}\hat{\Omega}$ have lengths $\mathrm{d}\xi$ and $\mathrm{d}\eta$, respectively. The vectors v_1 and v_2 are mapped to the vectors w_1 and w_2 with components

$$w_1 = \begin{pmatrix} \mathrm{d}x_1 \\ \mathrm{d}y_1 \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial \xi}\,\mathrm{d}\xi \\ \frac{\partial y}{\partial \xi}\,\mathrm{d}\xi \end{pmatrix}, \qquad w_2 = \begin{pmatrix} \mathrm{d}x_2 \\ \mathrm{d}y_2 \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial \eta}\,\mathrm{d}\eta \\ \frac{\partial y}{\partial \eta}\,\mathrm{d}\eta \end{pmatrix}.$$

Now, the area element $\mathrm{d}\hat{\Omega} = |v_1 \times v_2| = \mathrm{d}\xi\ \mathrm{d}\eta$, while the area element $\mathrm{d}\Omega = |w_1 \times w_2| = \det(\underline{J})\ \mathrm{d}\xi\ \mathrm{d}\eta$

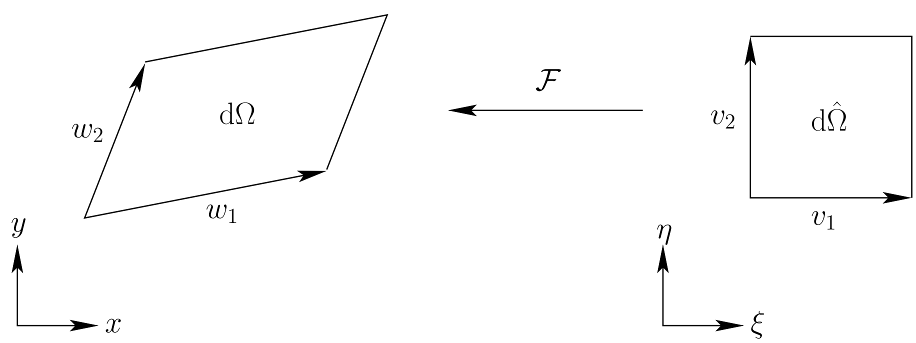


Figure 2: The surface element $\mathrm{d}\hat{\Omega}$ in $\hat{\Omega}$ is related to a corresponding surface element $\mathrm{d}\Omega$ in Ω via the mapping $\mathcal{F}$.

For example, if Ω is the rectangular domain $\Omega = (0, L_x) \times (0, L_y)$,

$$\underline{J} = \begin{pmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} \end{pmatrix} = \begin{pmatrix} \frac{L_x}{2} & 0 \\ 0 & \frac{L_y}{2} \end{pmatrix},$$

and

$$J = \det(\underline{J}) = \frac{L_x}{2}\frac{L_y}{2} = \frac{\mathrm{Area}(\Omega)}{\mathrm{Area}(\hat{\Omega})}.$$

In general,

$$\text{Area}(\Omega) = \int_\Omega \mathrm{d}\Omega = \int_{\hat{\Omega}} J \, \mathrm{d}\hat{\Omega}.$$

From (5) we have

$$\begin{pmatrix} \mathrm{d}x \\ \mathrm{d}y \end{pmatrix} = \underline{J} \begin{pmatrix} \mathrm{d}\xi \\ \mathrm{d}\eta \end{pmatrix}.$$

If we solve this system for $\mathrm{d}\xi$ and $\mathrm{d}\eta$, we obtain:

$$\begin{aligned} \mathrm{d}\xi &= \frac{1}{J}\left(\frac{\partial y}{\partial \eta}\,\mathrm{d}x - \frac{\partial x}{\partial \eta}\,\mathrm{d}y\right) \\ \mathrm{d}\eta &= \frac{1}{J}\left(\frac{\partial x}{\partial \xi}\,\mathrm{d}y - \frac{\partial y}{\partial \xi}\,\mathrm{d}x\right) \end{aligned}$$

or, in matrix form,

$$\begin{pmatrix} \mathrm{d}\xi \\ \mathrm{d}\eta \end{pmatrix} = \underline{J}^{-1} \begin{pmatrix} \mathrm{d}x \\ \mathrm{d}y \end{pmatrix}$$

with

$$\underline{J}^{-1} = \frac{1}{J}\begin{pmatrix} \frac{\partial y}{\partial \eta} & -\frac{\partial x}{\partial \eta} \\ -\frac{\partial y}{\partial \xi} & \frac{\partial x}{\partial \xi} \end{pmatrix}.$$

Alternatively, from (3) and (4), we also have

$$\begin{pmatrix} \mathrm{d}\xi \\ \mathrm{d}\eta \end{pmatrix} = \underbrace{\begin{pmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \xi}{\partial y} \\ \frac{\partial \eta}{\partial x} & \frac{\partial \eta}{\partial y} \end{pmatrix}}_{\underline{J}^{-1}} \begin{pmatrix} \mathrm{d}x \\ \mathrm{d}y \end{pmatrix}$$

Hence, we identify the very useful relationships

$$\frac{\partial \xi}{\partial x} = \frac{1}{J}\frac{\partial y}{\partial \eta}, \qquad \frac{\partial \xi}{\partial y} = -\frac{1}{J}\frac{\partial x}{\partial \eta} \tag{7}$$

$$\frac{\partial \eta}{\partial x} = -\frac{1}{J}\frac{\partial y}{\partial \xi}, \qquad \frac{\partial \eta}{\partial y} = \frac{1}{J}\frac{\partial x}{\partial \xi} \tag{8}$$

In the following, let us also use the abbreviated notation

$$\frac{\partial \xi}{\partial x} = \xi_x, \qquad \frac{\partial x}{\partial \eta} = x_\eta, \qquad \text{etc.}$$

Summary:

$$\underline{J} = \begin{pmatrix} x_\xi & x_\eta \\ y_\xi & y_\eta \end{pmatrix}, \qquad J = \det(\underline{J}) = x_\xi y_\eta - x_\eta y_\xi$$

$$\underline{J}^{-1} = \begin{pmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{pmatrix} = \frac{1}{J} \begin{pmatrix} y_\eta & -x_\eta \\ -y_\xi & x_\xi \end{pmatrix}$$

Exercise:
Rotate the rectangular domain $\Omega = (0, L_x) \times (0, L_y)$ with an angle θ. Verify that

$$\int_{\hat{\Omega}} J \,\mathrm{d}\xi \,\mathrm{d}\eta = L_x L_y = \mathrm{Area}(\Omega)$$

2 The Poisson problem in deformed geometries

Consider the two-dimensional Poisson problem:

$$\begin{aligned} -\nabla^2 u &= f \qquad \text{in } \Omega, \\ u &= 0 \qquad \text{on } \partial\Omega. \end{aligned}$$

where Ω is depicted in Figure 1. The corresponding weak form can be expressed as:
Find $u \in X = H_0^1(\Omega)$ such that

$$\int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x \,\mathrm{d}y = \int_\Omega f v \,\mathrm{d}x \,\mathrm{d}y \qquad \forall v \in X, \tag{9}$$

where

$$\nabla u \cdot \nabla v = \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y}.$$

Here,

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial \hat{u}}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial \hat{u}}{\partial \eta}\frac{\partial \eta}{\partial x} = \hat{u}_\xi \xi_x + \hat{u}_\eta \eta_x, \\ \frac{\partial u}{\partial y} &= \frac{\partial \hat{u}}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial \hat{u}}{\partial \eta}\frac{\partial \eta}{\partial y} = \hat{u}_\xi \xi_y + \hat{u}_\eta \eta_y, \end{aligned} \tag{10}$$

where we recall from earlier that $u(x, y) = u\big(x(\xi, \eta), y(\xi, \eta)\big) = \hat{u}(\xi, \eta)$. We get similar expressions for ∇v.

Let us denote the gradient of u in physical variables and in reference variables as

$$\nabla u = \begin{pmatrix} u_x \\ u_y \end{pmatrix} \quad \text{and} \quad \hat{\nabla}\hat{u} = \begin{pmatrix} \hat{u}_\xi \\ \hat{u}_\eta \end{pmatrix},$$

respectively. Using (10), we can relate ∇u and $\nabla \hat{u}$ through the relationship

$$\nabla u = \underline{G}_\nabla \hat{\nabla}\hat{u}, \tag{11}$$

where

$$\underline{G}_\nabla = \begin{pmatrix} \xi_x & \eta_x \\ \xi_y & \eta_y \end{pmatrix}. \tag{12}$$

Note that $\underline{G}_\nabla$ is not symmetric. Again, we get similar relationship for the gradient v.

We can now write $\nabla u \cdot \nabla v$ as

$$\begin{aligned} \nabla u \cdot \nabla v = (\nabla v)^T \nabla u &= (\underline{G}_\nabla \hat{\nabla}\hat{v})^T (\underline{G}_\nabla \hat{\nabla}\hat{u}) \\ &= (\hat{\nabla}\hat{v})^T (\underline{G}_\nabla^T \underline{G}_\nabla) \hat{\nabla}\hat{u} \\ &= (\hat{\nabla}\hat{v})^T \underline{G}\ \hat{\nabla}\hat{u}. \end{aligned} \tag{13}$$

Here, $\underline{G}$ is the symmetric matrix

$$\underline{G} = \underline{G}_\nabla^T \underline{G}_\nabla = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix},$$

with elements

$$\begin{aligned} g_{11} &= \xi_x^2 + \xi_y^2 = \left(\frac{y_\eta}{J}\right)^2 + \left(\frac{-x_\eta}{J}\right)^2 = \frac{1}{J^2}(y_\eta^2 + x_\eta^2) \\ g_{12} &= \xi_x \eta_x + \xi_y \eta_y = \frac{1}{J^2}(-y_\xi y_\eta - x_\eta x_\xi) \\ g_{21} &= g_{12} \\ g_{22} &= \eta_x^2 + \eta_y^2 = \frac{1}{J^2}(y_\xi^2 + x_\xi^2). \end{aligned}$$

We have here used (12) together with the relationships (7)-(8).

Using (13), we can express the weak form (9) as: Find $u \in X$ such that

$$\int_{\hat{\Omega}} (\hat{\nabla}\hat{v})^T \underline{G}\ \hat{\nabla}\hat{u}\, J \, \mathrm{d}\xi \, \mathrm{d}\eta = \int_{\hat{\Omega}} \hat{f}\, \hat{v}\, J \, \mathrm{d}\xi \, \mathrm{d}\eta \qquad \forall v \in X$$

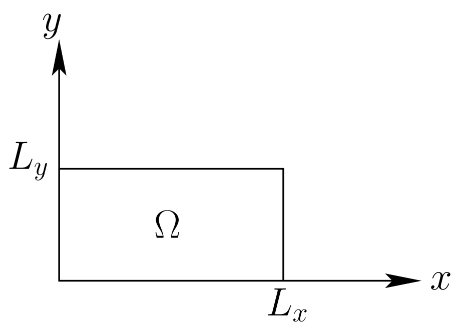
or

$$\int_{\hat{\Omega}} (\hat{\nabla}\hat{v})^T \underline{\tilde{G}}\ \hat{\nabla}\hat{u} \, \mathrm{d}\xi \, \mathrm{d}\eta = \int_{\hat{\Omega}} \hat{f}\, \hat{v}\, J \, \mathrm{d}\xi \, \mathrm{d}\eta \qquad \forall v \in X \tag{14}$$

where we have defined

$$\begin{aligned} \underline{\tilde{G}} &= J\underline{G} \\ \underline{\tilde{G}} &= \frac{1}{J}\begin{pmatrix} (x_\eta^2 + y_\eta^2) & -(x_\xi x_\eta + y_\xi y_\eta) \\ -(x_\xi x_\eta + y_\xi y_\eta) & (x_\xi^2 + y_\xi^2) \end{pmatrix}. \end{aligned} \tag{15}$$

In order to interpret the above transformation to reference variables, consider the simple undeformed case with $\Omega = (0, L_x) \times (0, L_y)$:



In this case

$$\underline{\tilde{G}} = \frac{1}{(\frac{L_x L_y}{4})}\begin{pmatrix} (\frac{L_y}{2})^2 & 0 \\ 0 & (\frac{L_x}{2})^2) \end{pmatrix} = \begin{pmatrix} \frac{L_y}{L_x} & 0 \\ 0 & \frac{L_x}{L_y} \end{pmatrix}$$

and hence,

$$(\hat{\nabla}\hat{v})^T \underline{\tilde{G}}\ \hat{\nabla}\hat{u} = \frac{L_y}{L_x}\frac{\partial \hat{u}}{\partial \xi}\frac{\partial \hat{v}}{\partial \xi} + \frac{L_x}{L_y}\frac{\partial \hat{u}}{\partial \eta}\frac{\partial \hat{v}}{\partial \eta}$$

which is the same result that we have derived earlier for undeformed geometries.

The general form for $\underline{\tilde{G}}$ allows us to handle undeformed geometries which are rotated, as well as general deformed geometries. All the results we have presented here extend readily to the three-dimensional case.

3 Computational approach

The discretization is based on the weak form (9) expressed using (14). In particular, the discrete problem that we want to solve is: Find $u_N \in X_N$ such that

$$\int_{\hat{\Omega}} (\hat{\nabla}\hat{v})^T \underline{\tilde{G}}\ \hat{\nabla}\hat{u}_N \, \mathrm{d}\xi \, \mathrm{d}\eta = \int_{\hat{\Omega}} \hat{f}\, \hat{v}\, J \, \mathrm{d}\xi \, \mathrm{d}\eta \qquad \forall v \in X_N. \tag{16}$$

Here, the discrete space ($X_N \subset X$) is chosen as

$$X_N = \{v \in H_0^1(\Omega) \quad | \quad v \circ \mathcal{F} \in \mathbb{P}_N(\hat{\Omega})\}.$$

In order to represent the geometry, we will use an *isoparametric* approach, meaning that we will approximate the geometry using the *same* order of expansion as for the solution, u_N, i.e.,

$$\begin{aligned} x_N(\xi, \eta) &\in \mathbb{P}_N(\hat{\Omega}), \\ y_N(\xi, \eta) &\in \mathbb{P}_N(\hat{\Omega}). \end{aligned}$$

Note that, in general, we cannot represent the geometry exactly, but will have to use some kind of approximation. If the surface of the domain Ω is piecewise smooth (smooth edges), and the domain is not too distorted, a high order approximation of the geometry will normally give very good results.

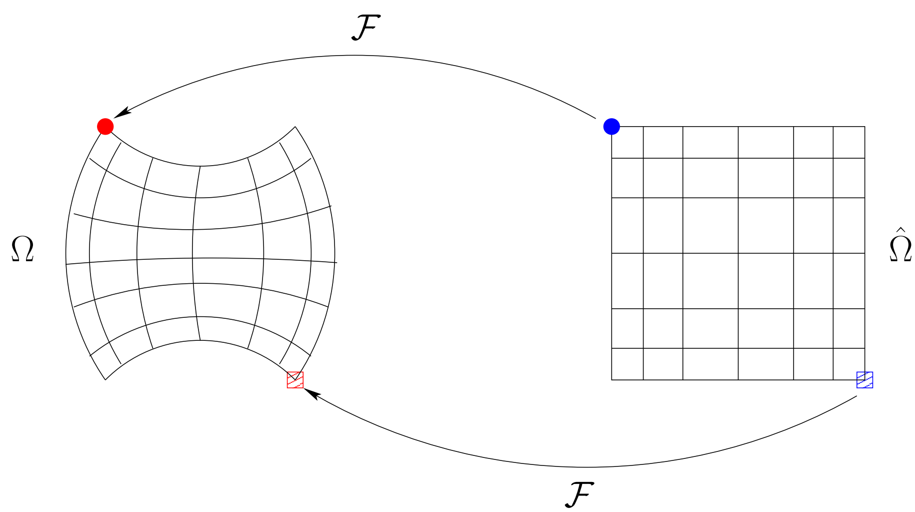
As a basis for our high order approximation, we again use a nodal, tensor-product, Lagrangian interpolant basis:

$$\begin{aligned} x_N(\xi, \eta) &= \sum_{i=0}^{N} \sum_{j=0}^{N} x_{ij} \ell_i(\xi) \ell_j(\eta), \\ y_N(\xi, \eta) &= \sum_{i=0}^{N} \sum_{j=0}^{N} y_{ij} \ell_i(\xi) \ell_j(\eta), \end{aligned}$$

with

$$\begin{aligned} \ell_i(\xi) &\in \mathbb{P}_N(-1, 1), \\ \ell_i(\xi_k) &= \delta_{ik}. \end{aligned}$$

The basis coefficients, $\{x_{ij}\}$ and $\{y_{ij}\}$, here represent the physical coordinates corresponding to the tensor-product Gauss-Lobatto Legendre (GLL) points under the mapping, $\mathcal{F}$; see Figure 3. Specifically, the GLL point (ξ_α, ξ_β) is mapped to the physical point $(x_N(\xi_\alpha, \xi_\beta), y_N(\xi_\alpha, \xi_\beta)) = (x_{\alpha\beta}, y_{\alpha\beta})$.



$$(x_{ij}, y_{ij}) \xleftarrow{\mathcal{F}} (\xi_i, \eta_j)$$

Figure 3: Each point in the reference domain is mapped to a unique point in the physical domain. In particular, the boundary of $\partial\widehat{\Omega}$ is mapped to $\partial\Omega$.

For example, the GLL points on the boundary of $\hat{\Omega}$ correspond to physical points on the boundary of Ω. In order to see how we compute the physical points corresponding to a GLL distribution on the reference domain, consider a simple example in Figure 4 with a polynomial degree $N = 4$ (i.e., 5 points in each spatial direction on $\hat{\Omega}$).

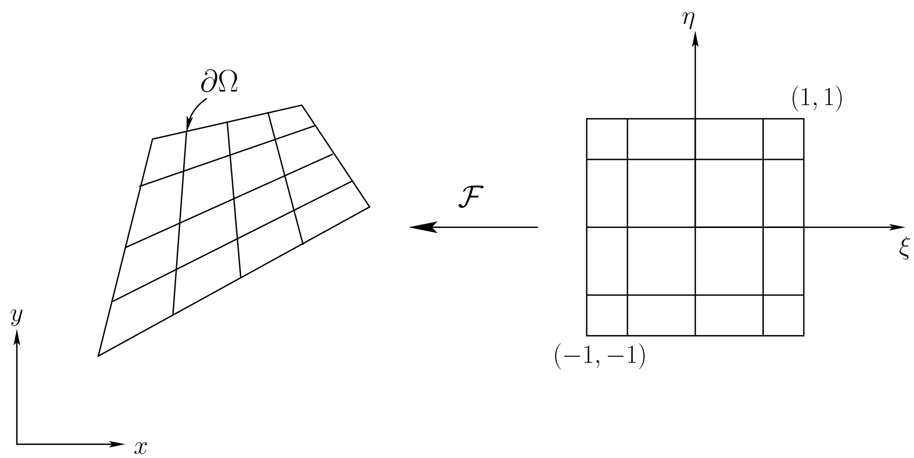


Figure 4: Each edge of the physical domain Ω is here a straight line.

Here, a GLL distribution of 5 points is first created along each edge of $\partial\Omega$. The internal physical points are then created by connecting the boundary points with straight lines (affine mapping). This will give us $\{x_{ij}\}$ and $\{y_{ij}\}$ for $0 \le i, j \le N$. We will return to the problem of generating $\{x_{ij}\}$ and $\{y_{ij}\}$ for general deformed geometries later.

Using a local data representation, define

$$\begin{aligned} \underline{X} &= \{x_{ij}\} \qquad \in \mathbb{R}^{(N+1)\times(N+1)} \\ \underline{Y} &= \{y_{ij}\} \qquad \in \mathbb{R}^{(N+1)\times(N+1)} \end{aligned}$$

Let us assume that we know $\{x_{ij}\}$ and $\{y_{ij}\}$ (or $\underline{\mathrm{X}}$ and $\underline{\mathrm{Y}}$). We can then compute

$$\begin{aligned} \left.\frac{\partial x_N}{\partial \xi}\right|_{(\xi_\alpha,\xi_\beta)} &= \sum_{i=0}^{N}\sum_{j=0}^{N} x_{ij} \underbrace{\ell_i'(\xi_\alpha)}_{D_{\alpha i}} \underbrace{\ell_j(\xi_\beta)}_{\delta_{\beta j}} \\ &= \sum_{i=0}^{N} D_{\alpha i} x_{i\beta} \qquad \forall \alpha, \beta \end{aligned}$$

or

$$\underline{X}_{,\xi} = \left\{ \frac{\partial x_N}{\partial \xi}(\xi_\alpha, \xi_\beta) \right\} = \underbrace{\underline{D}\ \underline{X}}_{\substack{\text{matrix-matrix product,} \\ \text{(e.g., using the BLAS library)}}}$$

Similarly,

$$\begin{aligned} \frac{\partial x_N}{\partial \eta}(\xi_\alpha, \xi_\beta) &= \sum_{i=0}^{N} \sum_{j=0}^{N} x_{ij} \underbrace{\ell_i(\xi_\alpha)}_{\delta_{\alpha i}} \underbrace{\ell_j'(\xi_\beta)}_{D_{\beta j}} \\ &= \sum_{j=0}^{N} D_{\beta j} x_{\alpha j} \end{aligned}$$

or

$$\underline{X}_{,\eta} = \left\{ \frac{\partial x_N}{\partial \eta}(\xi_\alpha, \xi_\beta) \right\} = \underbrace{\underline{X}\ \underline{D}^T}_{\substack{\text{matrix-matrix product.} \\ \mathcal{O}(N^3) \text{ operations}}}$$

In a similar fashion, we compute

$$\begin{aligned} \underline{Y}_{,\xi} = \left\{ \frac{\partial y_N}{\partial \xi}(\xi_\alpha, \xi_\beta) \right\} &= \underbrace{\underline{D}\ \underline{Y}}_{\substack{\text{matrix-matrix product.} \\ \mathcal{O}(N^3) \text{ operations}}} \\ \underline{Y}_{,\eta} = \left\{ \frac{\partial y_N}{\partial \eta}(\xi_\alpha, \xi_\beta) \right\} &= \underbrace{\underline{Y}\ \underline{D}^T}_{\substack{\text{matrix-matrix product.} \\ \mathcal{O}(N^3) \text{ operations}}} \end{aligned}$$

Next, we can compute

$$J_{\alpha\beta} = \left(\frac{\partial x_N}{\partial \xi} \frac{\partial y_N}{\partial \eta} - \frac{\partial x_N}{\partial \eta} \frac{\partial y_N}{\partial \xi} \right) \Bigg|_{(\xi_\alpha, \xi_\beta)} \qquad \forall \alpha, \beta$$

and

$$\underline{G}_{\alpha\beta} = \underline{G}\Big|_{(\xi_\alpha,\xi_\beta)} \qquad \text{(see Section 2)}$$

as well as

$$\underline{\tilde{G}}_{\alpha\beta} = (J\underline{G})\Big|_{(\xi_\alpha,\xi_\beta)} \qquad \text{in } \mathcal{O}(N^2) \text{ ops.}$$

3.1 Operator evaluation

We now return to the discrete problem (16), but this time we also include quadrature: Find $u_N \in X_N$ such that

$$\underbrace{\int_{\hat{\Omega}} (\hat{\nabla}\hat{v})^T \underline{\tilde{G}}\ \hat{\nabla}\hat{u}_N \,\mathrm{d}\xi\,\mathrm{d}\eta}_{\nwarrow} = \underbrace{\int_{\hat{\Omega}} \hat{f}\,\hat{v}\,J\,\mathrm{d}\xi\,\mathrm{d}\eta}_{\nearrow} \qquad \forall v \in X_N.$$

apply GLL quadrature

For example, the right hand side is approximated as

$$\int_{\hat{\Omega}} \hat{f}\,\hat{v}\,J\,\mathrm{d}\xi\,\mathrm{d}\eta \simeq \sum_{\alpha=0}^{N}\sum_{\beta=0}^{N} \rho_\alpha\rho_\beta f_{\alpha\beta}\, v_{\alpha\beta}\, J_{\alpha\beta}$$

We choose test functions $v \in X_N$ (recall that $v \circ \mathcal{F} = \hat{v}$) as: $\hat{v}(\xi,\eta) = \ell_i(\xi)\ell_j(\eta)$, $1 \le i,j \le N-1$, such that $v_{\alpha\beta} = \hat{v}(\xi_\alpha,\xi_\beta) = \delta_{i\alpha}\delta_{j\beta}$, in which case the right hand side reduces to

$$\int_{\hat{\Omega}} \hat{f}\,\hat{v}\,J\,\mathrm{d}\xi\,\mathrm{d}\eta \simeq \underbrace{(\rho_i\rho_j J_{ij})}_{\text{diagonal mass matrix}} f_{ij}, \qquad 1 \le i,j \le N-1.$$

Similarly, with

$$\hat{u}_N(\xi,\eta) = \sum_{m=1}^{N-1}\sum_{n=1}^{N-1} u_{mn}\ell_m(\xi)\ell_n(\eta),$$

we obtain for the left hand side:

$$\int_{\hat{\Omega}} (\hat{\nabla}v)^T \underline{\tilde{G}}\ \hat{\nabla}u_N \,\mathrm{d}\xi\,\mathrm{d}\eta \simeq \sum_{\alpha=0}^{N}\sum_{\beta=0}^{N} \rho_\alpha\rho_\beta \begin{pmatrix} D_{\alpha i}\delta_{\beta j} & \delta_{\alpha i}D_{\beta j}\end{pmatrix} \begin{pmatrix} (\tilde{g}_{11})_{\alpha\beta} & (\tilde{g}_{12})_{\alpha\beta} \\ (\tilde{g}_{21})_{\alpha\beta} & (\tilde{g}_{22})_{\alpha\beta}\end{pmatrix} \begin{pmatrix} u_{N,\xi} \\ u_{N,\eta}\end{pmatrix}_{(\xi_\alpha,\xi_\beta)}$$

The output of the above expression will be a set of nodal values, $\{w_{ij}\}$, which represent the action of the discrete Laplace operator upon u_N, with $v(\xi,\eta) = \ell_i(\xi)\ell_j(\eta), \quad \forall i,j$.

Let us now evaluate the last expression as a series of steps:

1) Compute $\underline{\tilde{G}}_{\alpha\beta}$ (preprocessing).

2) Compute

$$u_{N,\xi}\Big|_{(\xi_\alpha,\xi_\beta)} = D_{\alpha m} u_{m\beta} \qquad \mathcal{O}(N^3) \text{ ops.} \qquad \text{(matrix-matrix product)}$$

$$u_{N,\eta}\Big|_{(\xi_\alpha,\xi_\beta)} = u_{\alpha n} D^T_{n\beta} \qquad \mathcal{O}(N^3) \text{ ops.} \qquad \text{(matrix-matrix product)}$$

3) Compute

$$(s_1)_{\alpha\beta} = (\tilde{g}_{11})_{\alpha\beta} \cdot (u_{N,\xi})_{\alpha\beta} + (\tilde{g}_{12})_{\alpha\beta} \cdot (u_{N,\eta})_{\alpha\beta} \qquad \mathcal{O}(N^2) \text{ ops.}$$
$$(s_2)_{\alpha\beta} = (\tilde{g}_{21})_{\alpha\beta} \cdot (u_{N,\xi})_{\alpha\beta} + (\tilde{g}_{22})_{\alpha\beta} \cdot (u_{N,\eta})_{\alpha\beta} \qquad \mathcal{O}(N^2) \text{ ops.}$$

4) Compute

$$\left.\begin{aligned} (t_1)_{\alpha\beta} &= (s_1)_{\alpha\beta} \cdot \rho_\alpha \rho_\beta \\ (t_2)_{\alpha\beta} &= (s_2)_{\alpha\beta} \cdot \rho_\alpha \rho_\beta \end{aligned}\right\} \quad \mathcal{O}(N^2) \text{ ops.}$$

5)

$$w_{ij} = \underbrace{D^T_{i\alpha}(t_1)_{\alpha j}}_{\mathcal{O}(N^3)} + \underbrace{(t_2)_{i\beta} D_{\beta j}}_{\mathcal{O}(N^3)} \qquad \text{(matrix-matrix products)}$$

<u>Total work (excluding Step 1):</u>

$$\underbrace{2 \cdot 2N^3}_{\text{step 2}} + \underbrace{(3+3)N^2}_{\text{step 3}} + \underbrace{2N^2}_{\text{step 4}} + \underbrace{2 \cdot 2N^3 + N^2}_{\text{step 5}} = 8N^3 + 9N^2 = \underline{\mathcal{O}(N^3) \text{ ops.}}$$

In summary, starting from a set of nodal values $\underline{U}$ (with elements u_{mn}), we have just computed $\underline{W}$ (with elements w_{ij}) by acting on $\underline{U}$ with the discrete Laplace operator. This is an example of an operator evaluation, in this case, the evaluation of the bilinear form. The input is a field ($\underline{U}$); the output is a field ($\underline{W}$) representing the action on the input field by a discrete operator.

Note that, if we had used a global data representation ($\underline{u}$ and $\underline{w}$ instead of $\underline{U}$ and $\underline{W}$), we could have performed the above operator evaluation as $\underline{w} = \underline{A}\,\underline{u}$, where $\underline{A}$ is the discrete Laplace operator (explicit construction). Again, we prefer not to do this because of the larger computational cost and the larger memory requirement following such an approach.

4 Exercises

1. The computational cost for an operator evaluation with the discrete Laplace operator has a leading order term $8N^3$ in the general deformed case; see the last expression above. What is the leading order term for the undeformed case (i.e., a rectangle)?