Project I – Helmholtz equation in two dimensions

The deadline for this assignment is 28. October 2018 and counts towards 15% of the final grade. The delivery consists of a report answering all the questions and presenting the results, together with a source code in MATLAB. Working in pairs is possible but not compulsory.

1 Problem formulation

Let us consider the Helmholtz equation posed on $\Omega = (0,1)^2$,

$$u(\boldsymbol{x}) - \Delta u(\boldsymbol{x}) = f(\boldsymbol{x}), \qquad \forall\, \boldsymbol{x} = (x,y) \in \Omega \tag{1}$$

with source term $f(\boldsymbol{x}) = (2\pi^2 + 1)\cos(\pi x)\sin(\pi y)$, and supplemented with boundary conditions,

$$\begin{array}{lcccl} u(\boldsymbol{x}) & = & u_D & , & \boldsymbol{x} \in \Gamma_D = \{\boldsymbol{x} \in \partial\Omega : y \in \{0,1\}\} \\ \boldsymbol{\nabla} u \cdot \mathbf{n}(\boldsymbol{x}) & = & 0 & , & \boldsymbol{x} \in \Gamma_N = \{\boldsymbol{x} \in \partial\Omega : x \in \{0,1\}\} \end{array} \tag{2}$$

with $\mathbf{n}$ the outward normal to the boundary $\partial\Omega$.

The goal of this assignment is to write an algorithm to compute approximate solutions to Problem (1)–(2) using linear Lagrange Finite Elements on a triangular mesh $\mathcal{T}_h$.

(a) Verify that $\tilde{u} : (x,y) \mapsto \cos(\pi x)\sin(\pi y)$ is solution to (1)–(2).
(b) Derive a weak formulation of (1)–(2), specify the function spaces.
(c) Is the solution $\tilde{u}$ unique?

2 Finite Element space

The construction of the approximation space based on Lagrange $\mathbb{P}_1$ Finite Elements is now considered.

(a) Give the definition of the Lagrange $\mathbb{P}_1$ reference element on the unit triangle $\hat{K}$ with vertices $\{\hat{\mathrm{v}}_0 = (0,0), \hat{\mathrm{v}}_1 = (1,0), \hat{\mathrm{v}}_2 = (0,1)\}$, and associated local shape functions $(\hat{\varphi}_0, \hat{\varphi}_1, \hat{\varphi}_2)$, then implement it

(b) Write a simple test showing that shape functions $(\hat{\varphi}_0, \hat{\varphi}_1, \hat{\varphi}_2)$ form a nodal basis, and that for any $\hat{\boldsymbol{x}} \in \hat{K}$, $\hat{\varphi}_0(\hat{\boldsymbol{x}}) + \hat{\varphi}_1(\hat{\boldsymbol{x}}) + \hat{\varphi}_2(\hat{\boldsymbol{x}}) = 1$.

(c) Implement the affine mapping $T_K : \hat{K} \to K$ and verify that the determinant of the Jacobian J_{T_K} is positive for triangle $K_0 = \{(1,0),(3,1),(3,2)\}$. Interpret this result.

(d) Implement the inverse of J_{T_K} and verify that the Finite Element obtained by transporting the reference Finite Element $(\hat{K}, \hat{\mathcal{P}}, \hat{\Sigma})$ to K_0 is equivalent to $(\hat{K}, \hat{\mathcal{P}}, \hat{\Sigma})$.

(e) Formulate the Galerkin problem corresponding to the weak formulation derived at the preceding section, in particular define the approximation space carefully.

3 Numerical integration

Unless efficient exact evaluation is possible, computation of integrals is performed using quadrature rules. Such approximations are expressed as the weighted sum of integrand values over N_q *quadrature points* $\{\boldsymbol{\zeta}_q\}$,

$$I_K = \int_K \psi(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \approx \sum_{q=1}^{N_q} \psi(\boldsymbol{\zeta}_q)\,\omega_q$$

with real coefficients $\{\omega_q\}$ called *quadrature weights*. The order k_q of a quadrature rule is the polynomial degree of the integrand for which the evaluation is exact. In particular, Gauss–Legendre quadratures on a real interval gathered in Table 3 satisfy the relation $k_q = 2N_q - 1$. Quadrature rules can be defined using other polynomials and considering higher dimensions in space. In the frame of Finite Elements, contributions on the reference simplex $\hat{K}$ can be written as

$$\int_{\hat{K}} \hat{\psi}(\hat{\boldsymbol{x}})\,\mathrm{d}\hat{\boldsymbol{x}} \approx \sum_{q=1}^{N_q} \hat{\psi}(\boldsymbol{\zeta}_q)\,\hat{\omega}_q$$

with q the index of the quadrature point. Therefore any contribution on cell $K \in \mathcal{T}_h$ is obtained directly by composition with the affine change of coordinates T_K,

$$\int_K \psi(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \approx |\det(J_{T_K})| \sum_{q=1}^{N_q} \psi \circ T_K(\boldsymbol{\zeta}_q)\,\hat{\omega}_q$$

with J_{T_K} elementwise constant. If ψ involves derivatives, the change of variable should take into account that $(f \circ g)' = (f' \circ g) \cdot g'$.

(a) Implement Gauss–Legendre quadratures from Table 3 and plot the approximation error for

$$I = \int_1^2 e^x\,\mathrm{d}x$$

k_q	N_q	$\{\hat{\boldsymbol{\zeta}}_q\}$	$\{\hat{\omega}_q\}$
1	1	$\bar{\zeta}$	$\lvert I\rvert$
3	2	$\bar{\zeta} \pm \lvert I\rvert\frac{\sqrt{3}}{6}$	$\frac{1}{2}\lvert I\rvert$
5	3	$\bar{\zeta} \pm \lvert I\rvert\frac{\sqrt{15}}{10}$	$\frac{5}{18}\lvert I\rvert$
		$\bar{\zeta}$	$\frac{8}{18}\lvert I\rvert$
7	4	$\bar{\zeta} \pm \lvert I\rvert\frac{\sqrt{525+70\sqrt{30}}}{70}$	$\frac{18-\sqrt{30}}{36}\lvert I\rvert$
		$\bar{\zeta} \pm \lvert I\rvert\frac{\sqrt{525-70\sqrt{30}}}{70}$	$\frac{18+\sqrt{30}}{36}\lvert I\rvert$

Table 3: Gauss–Legendre quadratures on the interval $[a, b]$ with $\bar{\zeta} = (a+b)/2$, and $|I| = |b-a|$

k_q	N_q	$\{\hat{\boldsymbol{\zeta}}_q\}$	$\{\hat{\omega}_q\}$
1	1	$\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$	$\lvert K\rvert$
2	3	$\left(\frac{1}{2}, \frac{1}{2}, 0\right)_3$	$\frac{1}{3}\lvert K\rvert$
3	4	$\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$	$\frac{-9}{16}\lvert K\rvert$
		$\left(\frac{1}{5}, \frac{1}{5}, \frac{3}{5}\right)_3$	$\frac{25}{48}\lvert K\rvert$
4	7	$\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$	$\frac{9}{40}\lvert K\rvert$
		$(a_i, a_i, 1-2a_i)_3$	$\frac{155\pm\sqrt{15}}{1200}\lvert K\rvert$
		$a_i = \frac{6\pm\sqrt{15}}{21}$	

Table 3: Gauss–Legendre quadratures on a triangle K in barycentric coordinates $\{\hat{\boldsymbol{\zeta}}_q\} = (\lambda_0, \lambda_1, \lambda_2)$, with $(\cdot\,,\,\cdot\,,\,\cdot)_k$ the k distinct tuples obtained by permutation, [1] page 360.

(b) Implement Gauss–Legendre quadratures from Table 3 and plot the approximation error for

$$I = \int_{K_0} \log(x+y)\,\mathrm{d}x$$

with $K_0 = \{(1,0),(3,1),(3,2)\}$.

(c) Discuss why the choice of quadrature is important for the evaluation of Finite Element contributions. Which properties of the problem should be considered for terms corresponding to the left-hand side and right-hand side of the equation?

4 Assembly of the linear system

For each cell $K \in \mathcal{T}_h$, elementwise contributions for the Helmholtz equation are under the form of a sum of two submatrices, corresponding to contributions of the mass matrix

$$\mathrm{M}_K = \left[\int_K \varphi_j(\boldsymbol{x})\varphi_i(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\right]_{ij}$$

and of the stiffness matrix

$$\mathrm{K}_K = \left[\int_K \nabla\varphi_j(\hat{\boldsymbol{x}})\nabla\varphi_i(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\right]_{ij}$$

with j indices of the *global shape functions* (solution space), and i indices of the *global basis functions* (test space) with support on K. For any $K \in \mathcal{T}_h$ assembling the local equation consists of computing the contributions for indices $\hat{\jmath} = 1, \cdots, N_{\mathcal{P}}$ of the *local shape functions* (solution space), and $\hat{\imath} = 1, \cdots, N_{\mathcal{P}}$ indices of the *local basis functions* (test space) with support on K, $N_{\mathcal{P}}$ the dimension of the Finite Element. The passage from one to another is performed with a mapping from (cell) local indices $(\hat{\imath}, \hat{\jmath})$ to (mesh) global indices (i, j). The obtained submatrix and subvector are then added to the global matrix and load vector.

(a) Detail the assembly of the local matrix and the local vector for any $K \in \mathcal{T}_h$.
(b) Describe the assembly of the Dirichlet and Neumann boundary conditions.

5 Convergence analysis

(a) Implement the computation of the L^2 error norm given by

$$\|u - u_h\|_{\mathrm{L}^2} = \left(\int_\Omega |u(\boldsymbol{x}) - u_h(\boldsymbol{x})|^2\,\mathrm{d}\boldsymbol{x}\right)^{\frac{1}{2}}$$

Why should you be careful with the evaluation of the integral?

(b) Solve the problem for different mesh sizes $h_{\mathcal{T}} = 1/M$ with $M = 4, 8, 16$ and plot the L^2 error norm with respect to the dimension of the problem.

6 Extension to an evolution problem

Let us consider the evolution problem,

$$\partial_t u(\boldsymbol{x}, t) - \nu \Delta u(\boldsymbol{x}, t) = f(\boldsymbol{x}, t), \qquad \forall\ (\boldsymbol{x}, t) \in \Omega \times (0, T) \tag{3}$$

with $u(\boldsymbol{x}, 0) = u_0$ given initial data, and ν diffusivity.

(a) Describe how you would modify the algorithm developed for the Helmholtz problem to solve this equation for a given discretization in time. For example use the Backward Euler scheme. The function $\tilde{u}(\boldsymbol{x}, t) = e^{-\nu t} \sin\big(x \cos(\theta) + y \sin(\theta)\big)$ can be used to verify the implementation for the homogeneous equation (optional); take $\nu = 1$ and $\theta = \pi/4$ for instance.

Bibliography

[1] A. Ern and J.-L. Guermond. *Theory and Practice of Finite Elements*, volume 159 of *Springer Series: Applied Mathematical Sciences*. Springer, 2004.