

Project I – Helmholtz equation in two dimensions

The deadline for this assignment is 28. October 2018 and counts towards 15% of the final grade. The delivery consists of a report answering all the questions and presenting the results, together with a source code in MATLAB. Working in pairs is possible but not compulsory.

1 Problem formulation

Let us consider the Helmholtz equation posed on $\Omega = (0, 1)^2$,

$$u(\mathbf{x}) - \Delta u(\mathbf{x}) = f(\mathbf{x}), \quad \forall \mathbf{x} = (x, y) \in \Omega \quad (1)$$

with source term $f(\mathbf{x}) = (2\pi^2 + 1) \cos(\pi x) \sin(\pi y)$, and supplemented with boundary conditions,

$$\begin{aligned} u(\mathbf{x}) &= 0, & \mathbf{x} \in \Gamma_D &= \{\mathbf{x} \in \partial\Omega : y \in \{0, 1\}\} \\ \nabla u \cdot \mathbf{n}(\mathbf{x}) &= 0, & \mathbf{x} \in \Gamma_N &= \{\mathbf{x} \in \partial\Omega : x \in \{0, 1\}\} \end{aligned} \quad (2)$$

with \mathbf{n} the outward normal to the boundary $\partial\Omega$.

The goal of this assignment is to write an algorithm to compute approximate solutions to Problem (1)–(2) using linear Lagrange Finite Elements on a triangular mesh \mathcal{T}_h .

- Verify that $\tilde{u} : (x, y) \mapsto \cos(\pi x) \sin(\pi y)$ is solution to (1)–(2).
- Derive a weak formulation of (1)–(2), specify the function spaces.
- Is the solution \tilde{u} unique?

2 Finite Element space

The construction of the approximation space based on Lagrange \mathbb{P}_1 Finite Elements is now considered.

- Give the definition of the Lagrange \mathbb{P}_1 reference element on the unit triangle \hat{K} with vertices $\{\hat{\mathbf{v}}_0 = (0, 0), \hat{\mathbf{v}}_1 = (1, 0), \hat{\mathbf{v}}_2 = (0, 1)\}$, and associated local shape functions $(\hat{\varphi}_0, \hat{\varphi}_1, \hat{\varphi}_2)$, then implement it

- (b) Write a simple test showing that shape functions $(\hat{\varphi}_0, \hat{\varphi}_1, \hat{\varphi}_2)$ form a nodal basis, and that for any $\hat{\mathbf{x}} \in \hat{K}$, $\hat{\varphi}_0(\hat{\mathbf{x}}) + \hat{\varphi}_1(\hat{\mathbf{x}}) + \hat{\varphi}_2(\hat{\mathbf{x}}) = 1$.
- (c) Implement the affine mapping $T_K : \hat{K} \rightarrow K$ and verify that the determinant of the Jacobian J_{T_K} is positive for triangle $K_0 = \{(1, 0), (3, 1), (3, 2)\}$. Interpret this result.
- (d) Implement the inverse of J_{T_K} and verify that the Finite Element obtained by transporting the reference Finite Element $(\hat{K}, \hat{\mathcal{P}}, \hat{\Sigma})$ to K_0 is equivalent to $(\hat{K}, \hat{\mathcal{P}}, \hat{\Sigma})$.
- (e) Formulate the Galerkin problem corresponding to the weak formulation derived at the preceding section, in particular define the approximation space carefully.

3 Numerical integration

Unless efficient exact evaluation is possible, computation of integrals is performed using quadrature rules. Such approximations are expressed as the weighted sum of integrand values over N_q quadrature points $\{\zeta_q\}$,

$$I_K = \int_K \psi(\mathbf{x}) \, d\mathbf{x} \approx \sum_{q=1}^{N_q} \psi(\zeta_q) \omega_q$$

with real coefficients $\{\omega_q\}$ called *quadrature weights*. The order k_q of a quadrature rule is the polynomial degree of the integrand for which the evaluation is exact. In particular, Gauss–Legendre quadratures on a real interval gathered in Table 3 satisfy the relation $k_q = 2N_q - 1$. Quadrature rules can be defined using other polynomials and considering higher dimensions in space. In the frame of Finite Elements, contributions on the reference simplex \hat{K} can be written as

$$\int_{\hat{K}} \hat{\psi}(\hat{\mathbf{x}}) \, d\hat{\mathbf{x}} \approx \sum_{q=1}^{N_q} \hat{\psi}(\zeta_q) \hat{\omega}_q$$

with q the index of the quadrature point. Therefore any contribution on cell $K \in \mathcal{T}_h$ is obtained directly by composition with the affine change of coordinates T_K ,

$$\int_K \psi(\mathbf{x}) \, d\mathbf{x} \approx |\det(J_{T_K})| \sum_{q=1}^{N_q} \psi \circ T_K(\zeta_q) \hat{\omega}_q$$

with J_{T_K} elementwise constant. If ψ involves derivatives, the change of variable should take into account that $(f \circ g)' = (f' \circ g) \cdot g'$.

- (a) Implement Gauss–Legendre quadratures from Table 3 and plot the approximation error for

$$I = \int_1^2 e^x \, dx$$

k_q	N_q	$\{\hat{\zeta}_q\}$	$\{\hat{\omega}_q\}$
1	1	$\bar{\zeta}$	$ I $
3	2	$\bar{\zeta} \pm I \frac{\sqrt{3}}{6}$	$\frac{1}{2} I $
5	3	$\bar{\zeta} \pm I \frac{\sqrt{15}}{10}$	$\frac{5}{18} I $
		$\bar{\zeta}$	$\frac{8}{18} I $
7	4	$\bar{\zeta} \pm I \frac{\sqrt{525+70\sqrt{30}}}{70}$	$\frac{18-\sqrt{30}}{36} I $
		$\bar{\zeta} \pm I \frac{\sqrt{525-70\sqrt{30}}}{70}$	$\frac{18+\sqrt{30}}{36} I $

Table 3: Gauss–Legendre quadratures on the interval $[a, b]$ with $\bar{\zeta} = (a + b)/2$, and $|I| = |b - a|$

k_q	N_q	$\{\hat{\zeta}_q\}$	$\{\hat{\omega}_q\}$
1	1	$(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$	$ K $
2	3	$(\frac{1}{2}, \frac{1}{2}, 0)_3$	$\frac{1}{3} K $
3	4	$(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$	$\frac{-9}{16} K $
		$(\frac{1}{5}, \frac{1}{5}, \frac{3}{5})_3$	$\frac{25}{48} K $
4	6	$(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$	$\frac{9}{40} K $
		$(a_i, a_i, 1 - 2a_i)_3$	$\frac{155 \pm \sqrt{15}}{1200} K $
		$a_i = \frac{6 \pm \sqrt{15}}{21}$	

Table 3: Gauss–Legendre quadratures on a triangle K in barycentric coordinates $\{\hat{\zeta}_q\} = (\lambda_0, \lambda_1, \lambda_2)$, with $(\cdot, \cdot, \cdot)_k$ the k distinct tuples obtained by permutation, [1] page 360.

- (b) Implement Gauss–Legendre quadratures from Table 3 and plot the approximation error for

$$I = \int_{K_0} \log(x + y) \, dx$$

with $K_0 = \{(1, 0), (3, 1), (3, 2)\}$.

- (c) Discuss why the choice of quadrature is important for the evaluation of Finite Element contributions. Which properties of the problem should be considered for terms corresponding to the left-hand side and right-hand side of the equation?

4 Assembly of the linear system

For each cell $K \in \mathcal{T}_h$, elementwise contributions for the Helmholtz equation are under the form of a sum of two submatrices, corresponding to contributions of the mass matrix

$$M_K = \left[\int_K \varphi_j(\mathbf{x}) \varphi_i(\mathbf{x}) \, d\mathbf{x} \right]_{ij}$$

and of the stiffness matrix

$$K_K = \left[\int_K \nabla \varphi_j(\hat{\mathbf{x}}) \nabla \varphi_i(\mathbf{x}) \, d\mathbf{x} \right]_{ij}$$

with j indices of the *global shape functions* (solution space), and i indices of the *global basis functions* (test space) with support on K . For any $K \in \mathcal{T}_h$ assembling the local equation consists of computing the contributions for indices $\hat{j} = 1, \dots, N_{\mathcal{P}}$ of the *local shape functions* (solution space), and $\hat{i} = 1, \dots, N_{\mathcal{P}}$ indices of the *local basis functions* (test space) with support on K , $N_{\mathcal{P}}$ the dimension of the Finite Element. The passage from one to another is performed with a mapping from (cell) local indices (\hat{i}, \hat{j}) to (mesh) global indices (i, j) . The obtained submatrix and subvector are then added to the global matrix and load vector.

- (a) Detail the assembly of the local matrix and the local vector for any $K \in \mathcal{T}_h$.
 (b) Describe the assembly of the Dirichlet and Neumann boundary conditions.

5 Convergence analysis

- (a) Implement the computation of the L^2 error norm given by

$$\|u - u_h\|_{L^2} = \left(\int_{\Omega} |u(\mathbf{x}) - u_h(\mathbf{x})|^2 \, d\mathbf{x} \right)^{\frac{1}{2}}$$

Why should you be careful with the evaluation of the integral?

- (b) Solve the problem for different mesh sizes $h_{\mathcal{T}} = 1/M$ with $M = 4, 8, 16$ and plot the L^2 error norm with respect to the dimension of the problem.

6 Extension to an evolution problem

Let us consider the evolution problem,

$$\partial_t u(\mathbf{x}, t) - \nu \Delta u(\mathbf{x}, t) = f(\mathbf{x}, t), \quad \forall (\mathbf{x}, t) \in \Omega \times (0, T) \quad (3)$$

with $u(\mathbf{x}, 0) = u_0$ given initial data, and ν diffusivity.

- (a) Describe how you would modify the algorithm developed for the Helmholtz problem to solve this equation for a given discretization in time. For example use the Backward Euler scheme. The function $\tilde{u}(\mathbf{x}, t) = e^{-\nu t} \sin(x \cos(\theta) + y \sin(\theta))$ can be used to verify the implementation for the homogeneous equation (optional); take $\nu = 1$ and $\theta = \pi/4$ for instance.

Bibliography

- [1] A. Ern and J.-L. Guermond. *Theory and Practice of Finite Elements*, volume 159 of *Springer Series: Applied Mathematical Sciences*. Springer, 2004.