Chapter 3

Finite Element spaces

In the previous lectures we have studied the properties of coercive problems in an abstract setting and described Ritz and Galerkin methods for the approx imation of the solution to a PDE, respectively in the case of symmetric and non symmetric bilinear forms.

The abstract setting reads:

$$\left| \begin{array}{l} \text{Find } u_h \in V_h \subset \mathrm{H} \text{ such that:} \\ a(u_h, v_h) = L(v_h) \quad , \forall\, v_h \in V_h \end{array} \right.$$

such that:

- V_h is a finite dimensional approximation space characterized by a dis cretization parameter h,
- $a(\,\cdot\,,\,\cdot\,)$ is a continuous bilinear form on $V_h \times V_h$, coercive w.r.t $\|\cdot\|_V$,
- $L(\,\cdot\,)$ is a continuous linear form.

Under these assumptions existence and uniqueness of a solution to the ap proximate problem holds owing to the Lax Milgram Theorem and u_h is called discrete solution. Provided this abstract framework which allows us to seek approximate solutions to PDEs, we need to chose the approximate space V_h and construct a basis $\mathcal{B} = (\varphi_1, \cdots, \varphi_N)$ of V_h on which the discrete solution is decomposed:

$$u_h = \sum_{j=1}^{N_{V_h}} u_j \; \varphi_j$$

with $N_{V_h} = \dim(V_h)$, $\{u_j\}$ a family of N_{V_h} real numbers called *global degrees of freedom* and $\{\varphi_j\}$ a family of N_{V_h} elements of V_h called *global shape functions*.

To construct the approximate space V_h, we need two ingredients:

1. An admissible mesh $\mathcal{T}_h$ generated by a tesselation of domain Ω.
2. A reference finite element $(\hat{K}, \hat{\mathcal{P}}, \hat{\Sigma})$ to construct a basis of V_h.

3.1 Admissible mesh

Definition 3.1.1 (Mesh)**.** Let Ω be polygonal ($d = 2$) or polyhedral ($d = 3$) subset of $\mathbb{R}^d$, we define $\mathcal{T}_h$ (a triangulation in the simplicial case) as a finite family $\{K_i\}$ of disjoints non empty subsets of Ω named cells. Moreover $\mathcal{N}_h = \{\mathcal{N}_i\}$ denotes the set a vertices of $\mathcal{T}_h$ and $\varepsilon_h = \{\sigma_{KL} = K \cap L\}$ denotes the set of edges.

Definition 3.1.2 (Mesh size)**.**

$$h_{\mathcal{T}} = \max_{K \in \mathcal{T}_h} (\mathrm{diam}(K))$$

Definition 3.1.3 (Geometrically conforming mesh)**.** A mesh is said geometrically conforming if two neighbouring cells share either exactly one vertex, exactly one edge, or in the case $d = 3$ exactly one facet.

The meaning of the previous condition is that there should not be any "hanging node" on a facet. Moreover some theoretical results require that the mesh satisfies some regularity condition: for example, bounded ratio of equivalent ball diameter, Delaunay condition on the angles of a triangle, ...

3.2 Reference Finite Element

Definition 3.2.1 (Finite Element [4] page 19, [2] page 69)**.** A Finite Element consists of a triple $(K, \mathcal{P}, \Sigma)$, such that

- K is a compact, connected subset of $\mathbb{R}^d$ with non empty interior and with regular boundary (typically Lipshitz continuous),
- $\mathcal{P}$ is a finite dimensional vector space, $\mathrm{dim}(\mathcal{P}) = N$, of functions $p : K \to \mathbb{R}$, which is the space of shape functions,
- Σ is a set $\{\sigma\}_j$ of linear forms,

$$\begin{array}{rccl} \sigma_j : & \mathcal{P} & \to & \mathbb{R} \qquad\qquad , \forall\, j \in [\![1, N]\!] \\ & p & \mapsto & p_j = \sigma_j(p) \end{array}$$

  which is a basis of $\mathcal{L}(\mathcal{P}, \mathbb{R})$, the dual of $\mathcal{P}$.

Practically, the definition constructs first the Finite Element on a cell K which can be an interval ($d = 1$), a polygon ($d = 2$) or a polyhedron ($d = 3$) (Example: triangle, quadrangle, tetrahedron, hexahedron). Then an approximation space $\mathcal{P}$ (Example: polynomial space) and the local degrees of freedom Σ are chosen (Example: value at N geometrical nodes $\{a_i\}$, $\sigma_i(\varphi_j) = \varphi_j(a_i)$). The local shape functions $\{\varphi_i\}$ are then constructed so as to ensure unisolvence.

Proposition 3.2.2 (Determination of the local shape functions)**.** *Let* $\{\sigma_i\}_{1\leq i\leq N}$ *be the set of local degrees of freedoms, the local shape functions are defined as* $\{\varphi_i\}_{1\leq i\leq N}$ *a basis of* $\mathcal{P}$ *such that,*

$$\sigma_i(\varphi_j) = \delta_{ij} \quad , \forall\, i,j \in [\![1,N]\!]$$

Definition 3.2.3 (Unisolvence)**.** A Finite Element is said unisolvent if for any vector $(\alpha_1,\cdots,\alpha_N) \in \mathbb{R}^N$ there exists a unique representant $p \in \mathcal{P}$ such that $\sigma_i(p) = \alpha_i$, $\forall\ \in [\![1,N]\!]$.

The unisolvence property of a Finite Element is equivalent to construct Σ as dual basis of $\mathcal{P}$, thus we can express any function $p \in \mathcal{P}$ as

$$p = \sum_{j=1}^{N} \sigma_j(p)\ \varphi_j$$

the unique decomposition on $\{\varphi_j\}$, with $p_j = \sigma_j(p)$ the j th degree of freedom. In other words, the choice of $\Sigma = \{\sigma_j\}$ ensures that the vector of degree of freedoms $(p_1,\cdots,p_N)$ uniquely defines a function of $\mathcal{P}$. Defining Σ as dual basis of $\mathcal{P}$ is equivalent to:

$$\dim(\mathcal{P}) = \mathrm{card}(\Sigma) = N \tag{3.1a}$$

$$\forall\, p \in \mathcal{P},\ (\sigma_i(p) = 0, 1 \leq i \leq N) \Rightarrow (p = 0) \tag{3.1b}$$

in which Property (3.1a) ensures that Σ generates $\mathcal{L}(\mathcal{P},\mathbb{R})$ and Property (3.1b) that $\{\sigma_i\}$ are linearly independent.

Usually the unisolvence is part of the definition of a Finite Element since chosing the shape functions such that $\sigma_i(\varphi_j) = \delta_{ij}$ is equivalent.

Definition 3.2.4 (Local interpolation operator [4] page 20)**.**

$$\begin{array}{rccc} \pi_K : & V(K) & \rightarrow & \mathcal{P} \\ & v & \mapsto & \displaystyle\sum_{j=1}^{N} \sigma_j(v)\ \varphi_j \end{array}$$

Remark 3.2.5. The notation using the dual basis can be confusing but with the relation $\sigma_i(p) = p(a_i)$ in the nodal Finite Element case it is easier to under stand that the set Σ of linear forms defines how the interpolated function $\pi_h\, u$ "represents" its infinite dimensional counterpart u through the definition of the degrees of freedom. In the introduction, we defined simply $u_i = \sigma_i(u)$ without expliciting it. A natural choice is the pointwise representation $u_i = u(a_i)$ at geometrical nodes $\{a_i\}$, which is the case of Lagrange elements, but it is not the only possible choice ! For example, σ_i can be:

- a mean flux trough each facet of the element (Raviart Thomas)

$$\sigma_i(v) = \int_{\xi} v \cdot \mathbf{n}_{\xi}\ \mathrm{d}s$$

- a mean value over each facet of the element (Crouzeix Raviart)

$$\sigma_i(v) = \int_\xi v \,\mathrm{d}s$$

- a mean value of the tangential component over each facet of the element (Nédelec)

$$\sigma_i(v) = \int_\xi v \cdot \boldsymbol{\tau}_\xi \,\mathrm{d}s$$

A specific choice of linear form allows a control on a certain quantity: divergence for the first two examples, and curl for the third. The approximations will then not only be H^s conformal but also include the divergence or the curl in the space.

3.3 Transport of the Finite Element

In practice to avoid the construction of shape functions for any Finite Element $(K, \mathcal{P}, \Sigma)$, $K \in \mathcal{T}_h$, the local shape functions are evaluated for a *reference Finite Element* $(\hat{K}, \hat{\mathcal{P}}, \hat{\Sigma})$ defined on a *reference cell* $\hat{K}$ and then transported onto any cell K of the mesh. For example, in the case of simplicial meshes the reference cell in one dimension is the unit interval $[0, 1]$, in two dimension the unit triangle with vertices $\{(0, 0), (0, 1), (1, 0)\}$. In so doing, we can generate any Finite Element $(K, \mathcal{P}, \Sigma)$ on the mesh from $(\hat{K}, \hat{\mathcal{P}}, \hat{\Sigma})$ provided that we can construct a mapping such that $(K, \mathcal{P}, \Sigma)$ and $(\hat{K}, \hat{\mathcal{P}}, \hat{\Sigma})$ are equivalents.

Definition 3.3.1 (Equivalent Finite Elements)**.** Two Finite Elements $(K, \mathcal{P}, \Sigma)$ and $(\hat{K}, \hat{\mathcal{P}}, \hat{\Sigma})$ are said *equivalent* if there exists a bijection T_K from $\hat{K}$ onto K such that:

$$\forall\, p \in \mathcal{P},\ p \circ T_K \in \hat{\mathcal{P}}$$

and

$$\Sigma = T_K(\hat{\Sigma})$$

By collecting the local shape functions and local degrees of freedom from all the generated $(K, \mathcal{P}, \Sigma)$ on the mesh, we then construct *global shape functions* and *global degrees of freedom* and thus the approximation space V_h.

For Lagrange elements the transformation used to transport the Finite El ement on the mesh is an *affine mapping*, but this is not suitable in general !

3.4 Numerical integration

The contributions are integrated numerically, usually using quadrature rules.

3.5 Method

Algorithm 3.5.1 (Finite Element Method)**.** *Solving a problem by a Finite Element Method is defined by the following procedure:*

1. *Choose a reference finite element* $(\hat{K}, \hat{\mathcal{P}}, \hat{\Sigma})$.
2. *Construct an admissible mesh* $\mathcal{T}_h$ *such that any cell* $K \in \mathcal{T}_h$ *is in bijection with the reference cell* $\hat{K}$.
3. *Define a mapping to transport the reference finite element defined on* $\hat{K}$ *onto any* $K \in \mathcal{T}_h$ *to* $(K, \mathcal{P}, \Sigma)$.
4. *Construct a basis for* V_h *by collecting all the finite element basis of finite elements* $\{(K, \mathcal{P}, \Sigma)\}_{K \in \mathcal{T}_h}$ *sharing the same degree of freedom.*

Remark 3.5.2. The Finite Element approximation is said H conformal if $V_h \subset$ H and is said non conformal is $V_h \not\subset$ H. In this latter case the approximate problem can be constructed by building an approximate bilinear form

$$a_h(\,\cdot\,,\,\cdot\,) = a(\,\cdot\,,\,\cdot\,) + s(\,\cdot\,,\,\cdot\,)$$

as described, for instance, in the case of stabilized methods for advection dominated problems in Section **??**.

3.6 Exercises

Exercise 3.6.1.
Let us consider the Poisson problem posed on the domain $\Omega = (0, 1)$:

$$-u''(x) = f(x), \qquad \forall\, x \in \Omega \tag{3.2a}$$

with $f \in \mathrm{L}^2(\Omega)$, and satisfying the boundary condition on $\partial\Omega$

$$u(x) = 0, \qquad \forall\, x \in \partial\Omega \tag{3.2b}$$

The domain $\bar{\Omega}$ is discretized into a family of subintervals $[x_i, x_{i+1}]$, $i = 0, \cdots, N$, and Problem 3.2 is approximated by a linear Lagrange finite element method. The approximation space is the space of continuous piecewise linear functions $V_h = \{\varphi_i\}_{0\leq i\leq N}$ with

$$\varphi_i(x) = \begin{cases} \dfrac{x - x_{i-1}}{x_i - x_{i-1}}, & x_{i-1} \leq x \leq x_i,\ i \neq 0 \\ \dfrac{x_{i+1} - x}{x_{i+1} - x_i}, & x_i \leq x \leq x_{i+1},\ i \neq N \\ 0 \quad \textit{otherwise} \end{cases} \tag{3.3}$$

(a) Find the weak formulation of Problem 3.2.
(b) Prove that $a(u - u_h, \varphi_i) = 0$, for $i = 1, \cdots, N - 1$.
(c) Prove that for any $v \in \mathrm{H}^1([0, 1]) \cap \mathrm{C}^0([0, 1])$, $i = 1, \cdots, N - 1$:

$$a(v, \varphi_i) = \frac{1}{h}\big(-v(x_{i-1} + 2v(x_i) - v(x_{i+1})\big) \tag{3.4}$$

Let us consider $f(x) = x^4$:

(a) Find the expression of the solution to Problem 3.2.
(b) Give the expression of the linear system obtained by the suggested method on a uniform grid, *i.e.* $x_i = ih$, $i = 0, \cdots, N$.
(c) Implement a program computing the discrete solution u_h using the sug gested method.
(d) Plot the discrete solution u_h, the exact solution u, and the error $|u - u_h|$ with $N = 8, 16, 32$.
(e) Implement a function computing the L^2 error norm $\|u - u_h\|_{\mathrm{L}^2(\Omega)}$ and plot the value for different values of N.
(f) Modify the program to handle non uniform grids, given a list of node coordinates $\{x_i\}_{0\leq i\leq N}$.
(g) Based on the error $|u - u_h|$ suggest a distribution of the nodes $\{x_i\}_{0\leq i\leq N}$, repeat the same study, then compare the error values.