Chapter 6

Linear Elasticity

6.1 GENERALIZED HOOKE'S LAW. STRAIN ENERGY FUNCTION

In classical linear elasticity theory it is assumed that *displacements* and *displacement gradients* are sufficiently small that no distinction need be made between the Lagrangian and Eulerian descriptions. Accordingly in terms of the displacement vector u_i, the linear strain tensor is given by the equivalent expressions

$$l_{ij} = \epsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i}\right) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) = \tfrac{1}{2}(u_{i,j} + u_{j,i})$$

or

$$\mathbf{L} = \mathbf{E} = \tfrac{1}{2}(\mathbf{u}\nabla_{\mathbf{X}} + \nabla_{\mathbf{X}}\mathbf{u}) = \tfrac{1}{2}(\mathbf{u}\nabla_{\mathbf{x}} + \nabla_{\mathbf{x}}\mathbf{u}) = \tfrac{1}{2}(\mathbf{u}\nabla + \nabla\mathbf{u}) \qquad (6.1)$$

In the following it is further assumed that the deformation processes are *adiabatic* (no heat loss or gain) and *isothermal* (constant temperature) unless specifically stated otherwise.

The constitutive equations for a linear elastic solid relate the stress and strain tensors through the expression

$$\sigma_{ij} = C_{ijkm}\epsilon_{km} \quad \text{or} \quad \mathbf{\Sigma} = \tilde{\mathbf{C}}:\mathbf{E} \qquad (6.2)$$

which is known as the *generalized Hooke's law*. In (*6.2*) the *tensor of elastic constants* C_{ijkm} has 81 components. However, due to the symmetry of both the stress and strain tensors, there are at most 36 distinct elastic constants. For the purpose of writing Hooke's law in terms of these 36 components, the double indexed system of stress and strain components is often replaced by a single indexed system having a range of 6. Thus in the notation

$$\begin{aligned} \sigma_{11} &= \sigma_1 & \sigma_{23} &= \sigma_{32} = \sigma_4 \\ \sigma_{22} &= \sigma_2 & \sigma_{13} &= \sigma_{31} = \sigma_5 \\ \sigma_{33} &= \sigma_3 & \sigma_{12} &= \sigma_{21} = \sigma_6 \end{aligned} \qquad (6.3)$$

and

$$\begin{aligned} \epsilon_{11} &= \epsilon_1 & 2\epsilon_{23} &= 2\epsilon_{32} = \epsilon_4 \\ \epsilon_{22} &= \epsilon_2 & 2\epsilon_{13} &= 2\epsilon_{31} = \epsilon_5 \\ \epsilon_{33} &= \epsilon_3 & 2\epsilon_{12} &= 2\epsilon_{21} = \epsilon_6 \end{aligned} \qquad (6.4)$$

Hooke's law may be written

$$\sigma_K = C_{KM}\epsilon_M \quad (K, M = 1,2,3,4,5,6) \qquad (6.5)$$

where C_{KM} represents the 36 elastic constants, and where upper case Latin subscripts are used to emphasize the range of 6 on these indices.

When thermal effects are neglected, the energy balance equation (*5.32*) may be written

$$\frac{du}{dt} = \frac{1}{\rho}\sigma_{ij}D_{ij} = \frac{1}{\rho}\sigma_{ij}\dot{\epsilon}_{ij} \qquad (6.6)$$

The internal energy in this case is purely mechanical and is called the *strain energy* (per unit mass). From (*6.6*),

$$du = \frac{1}{\rho}\sigma_{ij}d\epsilon_{ij} \qquad (6.7)$$

and if u is considered a function of the nine strain components, $u = u(\epsilon_{ij})$, its differential is given by

$$du = \frac{\partial u}{\partial \epsilon_{ij}}d\epsilon_{ij} \qquad (6.8)$$

Comparing (*6.7*) and (*6.8*), it is observed that

$$\frac{1}{\rho}\sigma_{ij} = \frac{\partial u}{\partial \epsilon_{ij}} \qquad (6.9)$$

The *strain energy density* u^* (per unit volume) is defined as

$$u^* = \rho u \qquad (6.10)$$

and since ρ may be considered a constant in the small strain theory, u^* has the property that

$$\sigma_{ij} = \rho\frac{\partial u}{\partial \epsilon_{ij}} = \frac{\partial u^*}{\partial \epsilon_{ij}} \qquad (6.11)$$

Furthermore, the zero state of strain energy may be chosen arbitrarily; and since the stress must vanish with the strains, the simplest form of strain energy function that leads to a linear stress-strain relation is the quadratic form

$$u^* = \tfrac{1}{2}C_{ijkm}\epsilon_{ij}\epsilon_{km} \qquad (6.12)$$

From (*6.2*), this equation may be written

$$u^* = \tfrac{1}{2}\sigma_{ij}\epsilon_{ij} \quad \text{or} \quad u^* = \tfrac{1}{2}\mathbf{\Sigma}:\mathbf{E} \qquad (6.13)$$

In the single indexed system of symbols, (*6.12*) becomes

$$u^* = \tfrac{1}{2}C_{KM}\epsilon_K\epsilon_M \qquad (6.14)$$

in which $C_{KM} = C_{MK}$. Because of this symmetry on C_{KM}, the number of independent elastic constants is at most 21 if a strain energy function exists.

6.2 ISOTROPY. ANISOTROPY. ELASTIC SYMMETRY

If the elastic properties are independent of the reference system used to describe it, a material is said to be *elastically isotropic*. A material that is not isotropic is called *anisotropic*. Since the elastic properties of a Hookean solid are expressed through the coefficients C_{KM}, a general anisotropic body will have an *elastic-constant matrix* of the form

$$[C_{KM}] = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{21} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{31} & C_{32} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{41} & C_{42} & C_{43} & C_{44} & C_{45} & C_{46} \\ C_{51} & C_{52} & C_{53} & C_{54} & C_{55} & C_{56} \\ C_{61} & C_{62} & C_{63} & C_{64} & C_{65} & C_{66} \end{bmatrix} \qquad (6.15)$$

When a strain energy function exists for the body, $C_{KM} = C_{MK}$, and the 36 constants in (*6.15*) are reduced to 21.

A *plane of elastic symmetry* exists at a point where the elastic constants have the same values for every pair of coordinate systems which are the reflected images of one another with respect to the plane. The axes of such coordinate systems are referred to as "equivalent elastic directions." If the x_1x_2 plane is one of elastic symmetry, the constants C_{KM} are invariant under the coordinate transformation

$$x_1' = x_1, \quad x_2' = x_2, \quad x_3' = -x_3 \tag{6.16}$$

as shown in Fig. 6-1. The transformation matrix of (*6.16*) is given by

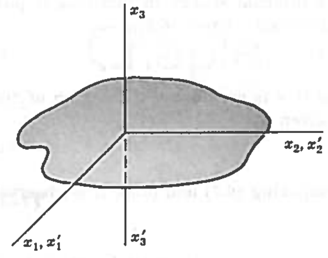


Fig. 6-1

$$[a_{ij}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \tag{6.17}$$

Inserting the values of (*6.17*) into the transformation laws for the linear stress and strain tensors, (*2.27*) and (*3.78*) respectively, the elastic matrix for a material having x_1x_2 as a plane of symmetry is

$$[C_{KM}] = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & C_{16} \\ C_{21} & C_{22} & C_{23} & 0 & 0 & C_{26} \\ C_{31} & C_{32} & C_{33} & 0 & 0 & C_{36} \\ 0 & 0 & 0 & C_{44} & C_{45} & 0 \\ 0 & 0 & 0 & C_{54} & C_{55} & 0 \\ C_{61} & C_{62} & C_{63} & 0 & 0 & C_{66} \end{bmatrix} \tag{6.18}$$

The 20 constants in (*6.18*) are reduced to 13 when a strain energy function exists.

If a material possesses three mutually perpendicular planes of elastic symmetry, the material is called *orthotropic* and its elastic matrix is of the form

$$[C_{KM}] = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{21} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{31} & C_{32} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix} \tag{6.19}$$

having 12 independent constants, or 9 if $C_{KM} = C_{MK}$.

An *axis of elastic symmetry* of order N exists at a point when there are sets of equivalent elastic directions which can be superimposed by a rotation through an angle of $2\pi/N$ about the axis. Certain cases of axial and plane elastic symmetry are equivalent.

6.3 ISOTROPIC MEDIA. ELASTIC CONSTANTS

Bodies which are elastically equivalent in all directions possess complete symmetry and are termed *isotropic*. Every plane and every axis is one of elastic symmetry in this case.

For isotropy, the number of independent elastic constants reduces to 2, and the elastic matrix is symmetric regardless of the existence of a strain energy function. Choosing as the two independent constants the well-known Lamé constants, λ and μ, the matrix (*6.19*) reduces to the isotropic elastic form

$$[C_{KM}] = \begin{bmatrix} \lambda+2\mu & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda+2\mu & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & \lambda+2\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu \end{bmatrix} \tag{6.20}$$

In terms of λ and μ, Hooke's law (*6.2*) for an isotropic body is written

$$\sigma_{ij} = \lambda\delta_{ij}\epsilon_{kk} + 2\mu\epsilon_{ij} \quad \text{or} \quad \boldsymbol{\Sigma} = \lambda \mathbf{I}\epsilon + 2\mu\mathbf{E} \tag{6.21}$$

where $\epsilon = \epsilon_{kk} = \mathrm{I}_{\mathbf{E}}$. This equation may be readily inverted to express the strains in terms of the stresses as

$$\epsilon_{ij} = \frac{-\lambda}{2\mu(3\lambda+2\mu)}\delta_{ij}\sigma_{kk} + \frac{1}{2\mu}\sigma_{ij} \quad \text{or} \quad \mathbf{E} = \frac{-\lambda}{2\mu(3\lambda+2\mu)}\mathbf{I}\Theta + \frac{1}{2\mu}\boldsymbol{\Sigma} \tag{6.22}$$

where $\Theta = \sigma_{kk} = \mathrm{I}_{\Sigma}$, the symbol traditionally used in elasticity for the first stress invariant.

For a simple uniaxial state of stress in the x_1 direction, engineering constants E and ν may be introduced through the relationships $\sigma_{11} = E\epsilon_{11}$ and $\epsilon_{22} = \epsilon_{33} = -\nu\epsilon_{11}$. The constant E is known as *Young's modulus*, and ν is called *Poisson's ratio*. In terms of these elastic constants Hooke's law for isotropic bodies becomes

$$\sigma_{ij} = \frac{E}{1+\nu}\left(\epsilon_{ij} + \frac{\nu}{1-2\nu}\delta_{ij}\epsilon_{kk}\right) \quad \text{or} \quad \boldsymbol{\Sigma} = \frac{E}{1+\nu}\left(\mathbf{E} + \frac{\nu}{1-2\nu}\mathbf{I}\epsilon\right) \tag{6.23}$$

or, when inverted,

$$\epsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\delta_{ij}\sigma_{kk} \quad \text{or} \quad \mathbf{E} = \frac{1+\nu}{E}\boldsymbol{\Sigma} - \frac{\nu}{E}\mathbf{I}\Theta \tag{6.24}$$

From a consideration of a uniform hydrostatic pressure state of stress, it is possible to define the *bulk modulus*,

$$K = \frac{E}{3(1-2\nu)} \quad \text{or} \quad K = \frac{3\lambda+2\mu}{3} \tag{6.25}$$

which relates the pressure to the cubical dilatation of a body so loaded. For a so-called state of pure shear, the *shear modulus* G relates the shear components of stress and strain. G is actually equal to μ and the expression

$$\mu = G = \frac{E}{2(1+\nu)} \tag{6.26}$$

may be proven without difficulty.

6.4 ELASTOSTATIC PROBLEMS. ELASTODYNAMIC PROBLEMS

In an elastostatic problem of a homogeneous isotropic body, certain field equations, namely,

(*a*) Equilibrium equations,

$$\sigma_{ji,j} + \rho b_i = 0 \quad \text{or} \quad \nabla\cdot\boldsymbol{\Sigma} + \rho\mathbf{b} = 0 \tag{6.27}$$

(*b*) Hooke's law,

$$\sigma_{ij} = \lambda\delta_{ij}\epsilon_{kk} + 2\mu\epsilon_{ij} \quad \text{or} \quad \Sigma = \lambda \mathbf{I}\epsilon + 2\mu\mathbf{E} \tag{6.28}$$

(*c*) Strain-displacement relations,

$$\epsilon_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}) \quad \text{or} \quad \mathbf{E} = \tfrac{1}{2}(\mathbf{u}\nabla + \nabla\mathbf{u}) \tag{6.29}$$

must be satisfied at all interior points of the body. Also, prescribed conditions on stress and/or displacements must be satisfied on the bounding surface of the body.

The boundary value problems of elasticity are usually classified according to boundary conditions into problems for which

(1) displacements are prescribed everywhere on the boundary,

(2) stresses (surface tractions) are prescribed everywhere on the boundary,

(3) displacements are prescribed over a portion of the boundary, stresses are prescribed over the remaining part.

For all three categories the body forces are assumed to be given throughout the continuum.

For those problems in which boundary displacement components are given everywhere by an equation of the form

$$u_i = g_i(\mathbf{X}) \quad \text{or} \quad \mathbf{u} = \mathbf{g}(\mathbf{X}) \tag{6.30}$$

the strain-displacement relations (*6.29*) may be substituted into Hooke's law (*6.28*) and the result in turn substituted into (*6.27*) to produce the governing equations,

$$\mu u_{i,jj} + (\lambda+\mu)u_{j,ji} + \rho b_i = 0 \quad \text{or} \quad \mu\nabla^2\mathbf{u} + (\lambda+\mu)\nabla\nabla\cdot\mathbf{u} + \rho\mathbf{b} = 0 \tag{6.31}$$

which are called the *Navier-Cauchy* equations. The solution of this type of problem is therefore given in the form of the displacement vector u_i, satisfying (*6.31*) throughout the continuum and fulfilling (*6.30*) on the boundary.

For those problems in which surface tractions are prescribed everywhere on the boundary by equations of the form

$$t_i^{(\hat{n})} = \sigma_{ij}n_j \quad \text{or} \quad \mathbf{t}^{(\hat{n})} = \Sigma\cdot\hat{\mathbf{n}} \tag{6.32}$$

the equations of compatibility (*3.104*) may be combined with Hooke's law (*6.24*) and the equilibrium equation (*6.27*) to produce the governing equations,

$$\sigma_{ij,kk} + \frac{1}{1+\nu}\sigma_{kk,ij} + \rho(b_{i,j} + b_{j,i}) + \frac{\nu}{1-\nu}\delta_{ij}\rho b_{k,k} = 0$$

or

$$\nabla^2\Sigma + \frac{1}{1+\nu}\nabla\nabla\Theta + \rho(\nabla\mathbf{b} + \mathbf{b}\nabla) + \frac{\nu}{1-\nu}\mathbf{I}\rho\nabla\cdot\mathbf{b} = 0 \tag{6.33}$$

which are called the *Beltrami-Michell* equations of compatibility. The solution for this type of problem is given by specifying the stress tensor which satisfies (*6.33*) throughout the continuum and fulfills (*6.32*) on the boundary.

For those problems having "mixed" boundary conditions, the system of equations (*6.27*), (*6.28*) and (*6.29*) must be solved. The solution gives the stress and displacement fields throughout the continuum. The stress components must satisfy (*6.32*) over some portion of the boundary, while the displacements satisfy (*6.30*) over the remainder of the boundary.

In the formulation of elastodynamics problems, the equilibrium equations (*6.27*) must be replaced by the equations of motion (*5.16*)

$$\sigma_{ij,j} + \rho b_i = \rho\dot{v}_i \quad \text{or} \quad \nabla\cdot\Sigma + \rho\mathbf{b} = \rho\dot{\mathbf{v}} \tag{6.34}$$

and *initial conditions* as well as *boundary conditions* must be specified. In terms of the displacement field u_i, the governing equation here, analogous to (*6.31*) in the elastostatic case is

$$\mu u_{i,jj} + (\lambda+\mu)u_{j,ji} + \rho b_i = \rho\ddot{u}_i \quad \text{or} \quad \mu\nabla^2\mathbf{u} + (\lambda+\mu)\nabla\nabla\cdot\mathbf{u} + \rho\mathbf{b} = \rho\ddot{\mathbf{u}} \tag{6.35}$$

Solutions of (*6.35*) appear in the form $u_i = u_i(\mathbf{x}, t)$ and must satisfy not only initial conditions on the motion, usually expressed by equations such as

$$u_i = u_i(\mathbf{x}, 0) \quad \text{and} \quad \dot{u}_i = \dot{u}_i(\mathbf{x}, 0) \tag{6.36}$$

but also boundary conditions, either on the displacements,

$$u_i = g_i(\mathbf{x}, t) \quad \text{or} \quad \mathbf{u} = \mathbf{g}(\mathbf{x}, t) \tag{6.37}$$

or on the surface tractions,

$$t_i^{(\hat{n})} = t_i^{(\hat{n})}(\mathbf{x}, t) \quad \text{or} \quad \mathbf{t}^{(\hat{n})} = \mathbf{t}^{(\hat{n})}(\mathbf{x}, t) \tag{6.38}$$

6.5 THEOREM OF SUPERPOSITION. UNIQUENESS OF SOLUTIONS. ST. VENANT PRINCIPLE

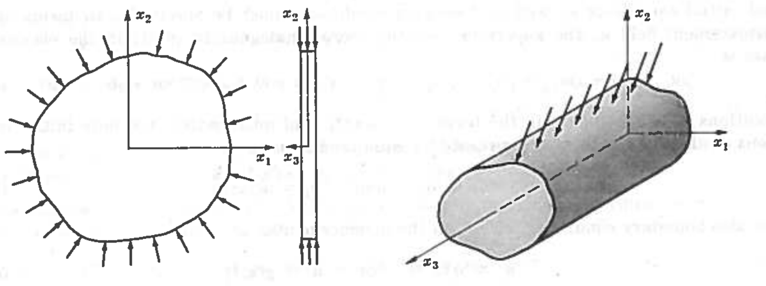
Because the equations of linear elasticity are linear equations, the principle of superposition may be used to obtain additional solutions from those previously established. If, for example, $\sigma_{ij}^{(1)}, u_i^{(1)}$ represent a solution to the system (*6.27*), (*6.28*) and (*6.29*) with body forces $b_i^{(1)}$, and $\sigma_{ij}^{(2)}, u_i^{(2)}$ represent a solution for body forces $b_i^{(2)}$, then $\sigma_{ij} = \sigma_{ij}^{(1)} + \sigma_{ij}^{(2)}$, $u_i = u_i^{(1)} + u_i^{(2)}$ represent a solution to the system for body forces $b_i = b_i^{(1)} + b_i^{(2)}$.

The uniqueness of a solution to the general elastostatic problem of elasticity may be established by use of the superposition principle, together with the law of conservation of energy. A proof of uniqueness is included among the exercises that follow.

St. Venant's principle is a statement regarding the differences that occur in the stresses and strains at some interior location of an elastic body, due to two separate but statically equivalent systems of surface tractions, being applied to some portion of the boundary. The principle asserts that, for locations sufficiently remote from the area of application of the loadings, the differences are negligible. This assumption is often of great assistance in solving practical problems.

6.6 TWO-DIMENSIONAL ELASTICITY. PLANE STRESS AND PLANE STRAIN

Many problems in elasticity may be treated satisfactorily by a two-dimensional, or *plane theory of elasticity*. There are two general types of problems involved in this plane analysis. Although these two types may be defined by setting down certain restrictions and assumptions on the stress and displacement fields, they are often introduced descriptively in terms of their physical prototypes. In *plane stress* problems, the geometry of the body is essentially that of a plate with one dimension much smaller than the others. The loads are applied uniformly over the thickness of the plate and act in the plane of the plate as shown in Fig. 6-2(*a*) below. In *plane strain* problems, the geometry of the body is essentially that of a prismatic cylinder with one dimension much larger than the others. The loads are uniformly distributed with respect to the large dimension and act perpendicular to it as shown in Fig. 6-2(*b*) below.



(a) (b)

Fig. 6-2

For the plane stress problem of Fig. 6-2(a) the stress components $\sigma_{33}, \sigma_{13}, \sigma_{23}$ are taken as zero everywhere, and the remaining components are taken as functions of x_1 and x_2 only,

$$\sigma_{\alpha\beta} = \sigma_{\alpha\beta}(x_1, x_2) \qquad (\alpha, \beta = 1, 2) \tag{6.39}$$

Accordingly, the field equations for plane stress are

(a) $$\sigma_{\alpha\beta,\beta} + \rho b_\alpha = 0 \quad \text{or} \quad \nabla \cdot \Sigma + \rho \mathbf{b} = 0 \tag{6.40}$$

(b) $$\epsilon_{\alpha\beta} = \frac{1+\nu}{E}\sigma_{\alpha\beta} - \frac{\nu}{E}\delta_{\alpha\beta}\sigma_{\gamma\gamma} \quad \text{or} \quad \mathbf{E} = \frac{1+\nu}{E}\Sigma - \frac{\nu}{E}\mathbf{I}\Theta$$
$$\epsilon_{33} = -\frac{\nu}{E}\sigma_{\alpha\alpha} \tag{6.41}$$

(c) $$\epsilon_{\alpha\beta} = \tfrac{1}{2}(u_{\alpha,\beta} + u_{\beta,\alpha}) \quad \text{or} \quad \mathbf{E} = \tfrac{1}{2}(\mathbf{u}\nabla + \nabla\mathbf{u}) \tag{6.42}$$

in which $\nabla \equiv \frac{\partial}{\partial x_1}\hat{\mathbf{e}}_1 + \frac{\partial}{\partial x_2}\hat{\mathbf{e}}_2$ and

$$\Sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} & 0 \\ \sigma_{12} & \sigma_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \mathbf{E} = \begin{pmatrix} \epsilon_{11} & \epsilon_{12} & 0 \\ \epsilon_{12} & \epsilon_{22} & 0 \\ 0 & 0 & \epsilon_{33} \end{pmatrix} \tag{6.43}$$

Due to the particular form of the strain tensor in the plane stress case, the six compatibility equations (3.104) may be reduced with reasonable accuracy for very thin plates to the single equation

$$\epsilon_{11,22} + \epsilon_{22,11} = 2\epsilon_{12,12} \tag{6.44}$$

In terms of the displacement components u_α, the field equations may be combined to give the governing equation

$$\frac{E}{2(1+\nu)}\nabla^2 u_\alpha + \frac{E}{2(1-\nu)}u_{\beta,\beta\alpha} + \rho b_\alpha = 0 \quad \text{or} \quad \frac{E}{2(1+\nu)}\nabla^2\mathbf{u} + \frac{E}{2(1-\nu)}\nabla\nabla\cdot\mathbf{u} + \rho\mathbf{b} = 0 \tag{6.45}$$

where $\nabla^2 \equiv \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$.

For the plane strain problem of Fig. 6-2(b) the displacement component u_3 is taken as zero, and the remaining components considered as functions of x_1 and x_2 only,

$$u_\alpha = u_\alpha(x_1, x_2) \tag{6.46}$$

In this case, the field equations may be written

(a) $$\sigma_{\alpha\beta,\beta} + \rho b_\alpha = 0 \quad \text{or} \quad \nabla\cdot\Sigma + \rho\mathbf{b} = 0 \tag{6.47}$$

(b) $$\sigma_{\alpha\beta} = \lambda\delta_{\alpha\beta}\epsilon_{\gamma\gamma} + 2\mu\epsilon_{\alpha\beta} \quad \text{or} \quad \Sigma = \lambda\mathbf{I}\epsilon + 2\mu\mathbf{E}$$
$$\sigma_{33} = \nu\sigma_{\alpha\alpha} = \frac{\lambda}{2(\lambda+\mu)}\sigma_{\alpha\alpha} \tag{6.48}$$

(c) $$\epsilon_{\alpha\beta} = \tfrac{1}{2}(u_{\alpha,\beta} + u_{\beta,\alpha}) \quad \text{or} \quad \mathbf{E} = \tfrac{1}{2}(\mathbf{u}\nabla + \nabla\mathbf{u}) \tag{6.49}$$

in which $$\Sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} & 0 \\ \sigma_{12} & \sigma_{22} & 0 \\ 0 & 0 & \sigma_{33} \end{pmatrix} \quad \text{and} \quad \mathbf{E} = \begin{pmatrix} \epsilon_{11} & \epsilon_{12} & 0 \\ \epsilon_{12} & \epsilon_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{6.50}$$

From (6.47), (6.48), (6.49), the appropriate Navier equation for plane strain is

$$\mu\nabla^2 u_\alpha + (\lambda+\mu)u_{\beta,\beta\alpha} + \rho b_\alpha = 0 \quad \text{or} \quad \mu\nabla^2\mathbf{u} + (\lambda+\mu)\nabla\nabla\cdot\mathbf{u} + \rho\mathbf{b} = 0 \tag{6.51}$$

As in the case of plane stress, the compatibility equations for plane strain reduce to the single equation (6.44).

If the forces applied to the edge of the plate in Fig. 6-2(a) are not uniform across the thickness, but are symmetrical with respect to the middle plane of the plate, a state of *generalized plane stress* is said to exist. In formulating problems for this case, the field variables $\sigma_{\alpha\beta}$, $\epsilon_{\alpha\beta}$ and u_α must be replaced by stress, strain and displacement variables averaged across the thickness of the plate. In terms of such averaged field variables, the generalized plane stress formulation is essentially the same as the plane strain case if λ is replaced by

$$\lambda' = \frac{2\lambda\mu}{\lambda + 2\mu} = \frac{\nu E}{1-\nu^2} \tag{6.52}$$

A case of *generalized plane strain* is sometimes mentioned in elasticity books when ϵ_{33} is taken as a constant other than zero in (6.50).

6.7 AIRY'S STRESS FUNCTION

If body forces are absent or are constant, the solution of *plane elastostatic problems* (plane strain or generalized plane stress problems) is often obtained through the use of the *Airy stress function.* Even if body forces must be taken into account, the superposition principle allows for their contribution to the solution to be introduced as a particular integral of the linear differential field equations.

For plane elastostatic problems in the absence of body forces, the equilibrium equations reduce to

$$\sigma_{\alpha\beta,\beta} = 0 \quad \text{or} \quad \nabla\cdot\Sigma = 0 \tag{6.53}$$

and the compatibility equation (6.44) may be expressed in terms of stress components as

$$\nabla^2(\sigma_{11} + \sigma_{22}) = 0, \qquad \nabla^2\Theta_1 = 0 \tag{6.54}$$

The stress components are now given as partial derivatives of the Airy stress function $\phi = \phi(x_1, x_2)$ in accordance with the equations

$$\sigma_{11} = \phi_{,22}, \quad \sigma_{12} = -\phi_{,12}, \quad \sigma_{22} = \phi_{,11} \tag{6.55}$$

The equilibrium equations (*6.53*) are satisfied identically, and the compatibility condition (*6.54*) becomes the *biharmonic equation*

$$\nabla^2(\nabla^2\phi) = \nabla^4\phi = \phi_{,1111} + 2\phi_{,1122} + \phi_{,2222} = 0 \tag{6.56}$$

Functions which satisfy (*6.56*) are called *biharmonic functions*. By considering biharmonic functions with single-valued second partial derivatives, numerous solutions to plane elastostatic problems may be constructed, which satisfy automatically both equilibrium and compatibility. Of course these solutions must be tailored to fit whatever boundary conditions are prescribed.

6.8 TWO-DIMENSIONAL ELASTOSTATIC PROBLEMS IN POLAR COORDINATES

Body geometry often deems it convenient to formulate two-dimensional elastostatic problems in terms of polar coordinates r and θ. Thus for transformation equations

$$x_1 = r\cos\theta, \qquad x_2 = r\sin\theta \tag{6.57}$$

the stress components shown in Fig. 6-3 are found to lead to equilibrium equations in the form

$$\frac{\partial\sigma_{(rr)}}{\partial r} + \frac{1}{r}\frac{\partial\sigma_{(r\theta)}}{\partial\theta} + \frac{\sigma_{(rr)} - \sigma_{(\theta\theta)}}{r} + R = 0 \tag{6.58}$$

$$\frac{1}{r}\frac{\partial\sigma_{(\theta\theta)}}{\partial\theta} + \frac{\partial\sigma_{(r\theta)}}{\partial r} + \frac{2\sigma_{(r\theta)}}{r} + Q = 0 \tag{6.59}$$

in which R and Q represent body forces per unit volume in the directions shown.

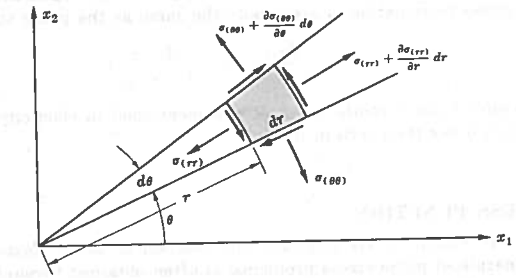


Fig. 6-3

Taking the Airy stress function now as $\Phi = \Phi(r,\theta)$, the stress components are given by

$$\sigma_{(rr)} = \frac{1}{r}\frac{\partial\Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2} \tag{6.60}$$

$$\sigma_{(\theta\theta)} = \partial^2\Phi/\partial r^2 \tag{6.61}$$

$$\sigma_{(r\theta)} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\Phi}{\partial\theta}\right) \tag{6.62}$$

The compatibility condition again leads to the biharmonic equation

$$\nabla^2(\nabla^2\Phi) = \nabla^4\Phi = 0 \tag{6.63}$$

but, in polar form, $\nabla^2 = \dfrac{\partial^2}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2}{\partial\theta^2}$.

6.9 HYPERELASTICITY. HYPOELASTICITY

Modern continuum studies have led to constitutive equations which define materials that are elastic in a special sense. In this regard a material is said to be *hyperelastic* if it possesses a strain energy function U such that the material derivative of this function is equal to the stress power per unit volume. Thus the constitutive equation is of the form

$$\frac{d}{dt}(U) = \frac{1}{\rho}\sigma_{ij}D_{ij} = \frac{1}{\rho}\sigma_{ij}\dot{\epsilon}_{ij} \tag{6.64}$$

in which D_{ij} is the rate of deformation tensor. In a second classification, a material is said to be *hypoelastic* if the stress rate is a homogeneous linear function of the rate of deformation. In this case the constitutive equation is written

$$\sigma^{\nabla}_{ij} = K_{ijkm}D_{km} \tag{6.65}$$

in which the stress rate σ^{∇}_{ij} is defined as

$$\sigma^{\nabla}_{ij} = \frac{d}{dt}(\sigma_{ij}) - \sigma_{iq}V_{qj} - \sigma_{jq}V_{qi} \tag{6.66}$$

where V_{ij} is the vorticity tensor.

6.10 LINEAR THERMOELASTICITY

If thermal effects are taken into account, the components of the linear strain tensor ϵ_{ij} may be considered to be the sum

$$\epsilon_{ij} = \epsilon^{(S)}_{ij} + \epsilon^{(T)}_{ij} \tag{6.67}$$

in which $\epsilon^{(S)}_{ij}$ is the contribution from the stress field and $\epsilon^{(T)}_{ij}$ is the contribution from the temperature field. Due to a change from some reference temperature T_0 to the temperature T, the strain components of an elementary volume of an unconstrained isotropic body are given by

$$\epsilon^{(T)}_{ij} = \alpha(T - T_0)\delta_{ij} \tag{6.68}$$

where α denotes the linear coefficient of thermal expansion. Inserting (*6.68*), together with Hooke's law (*6.22*), into (*6.67*) yields

$$\epsilon_{ij} = \frac{1}{2\mu}\left(\sigma_{ij} - \frac{\lambda}{3\lambda + 2\mu}\delta_{ij}\sigma_{kk}\right) + \alpha(T - T_0)\delta_{ij} \tag{6.69}$$

which is known as the *Duhamel-Neumann* relations. Equation (*6.69*) may be inverted to give the thermoelastic constitutive equations

$$\sigma_{ij} = \lambda\delta_{ij}\epsilon_{kk} + 2\mu\epsilon_{ij} - (3\lambda + 2\mu)\alpha\delta_{ij}(T - T_0) \tag{6.70}$$

Heat conduction in an isotropic elastic solid is governed by the well-known *Fourier law* of heat conduction,

$$c_i = -kT_{,i} \tag{6.71}$$

where the scalar k, the thermal conductivity of the body, must be positive to assure a positive rate of entropy production. If now the *specific heat* at constant deformation $c^{(v)}$ is introduced through the equation

$$-c_{i,i} = \rho c^{(v)}\dot{T} \tag{6.72}$$

and the internal energy is assumed to be a function of the strain components ϵ_{ij} and the temperature T, the energy equation (*5.45*) may be expressed in the form

$$kT_{,ii} = \rho c^{(v)}\dot{T} + (3\lambda + 2\mu)\alpha T_0 \dot{\epsilon}_{ii} \qquad (6.73)$$

which is known as the *coupled heat equation*.

The system of equations that formulate the general thermoelastic problem for an isotropic body consists of

(*a*) equations of motion

$$\sigma_{ij,j} + \rho b_i = \ddot{u}_i \quad \text{or} \quad \nabla \cdot \mathbf{\Sigma} + \rho \mathbf{b} = \ddot{\mathbf{u}} \qquad (6.74)$$

(*b*) thermoelastic constitutive equations

$$\sigma_{ij} = \lambda \delta_{ij}\epsilon_{kk} + 2\mu\epsilon_{ij} - (3\lambda + 2\mu)\alpha\delta_{ij}(T - T_0)$$

or $\qquad (6.75)$

$$\mathbf{\Sigma} = \lambda \mathbf{I}\epsilon + 2\mu \mathbf{E} - (3\lambda + 2\mu)\alpha \mathbf{I}(T - T_0)$$

(*c*) strain-displacement relations

$$\epsilon_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}) \quad \text{or} \quad \mathbf{E} = \tfrac{1}{2}(\mathbf{u}\nabla + \nabla\mathbf{u}) \qquad (6.76)$$

(*d*) coupled heat equation

$$kT_{,ii} = \rho c^{(v)}\dot{T} + (3\lambda + 2\mu)\alpha T_0 \dot{\epsilon}_{kk} \quad \text{or} \quad k\nabla^2 T = \rho c^{(v)}\dot{T} + (3\lambda + 2\mu)\alpha T_0 \dot{\epsilon} \qquad (6.77)$$

This system must be solved for the stress, displacement and temperature fields, subject to appropriate initial and boundary conditions. In addition, the compatibility equations must be satisfied.

There is a large collection of problems in which both the inertia and coupling effects may be neglected. For these cases the general thermoelastic problem decomposes into two separate problems which must be solved consecutively, but independently. Thus for the uncoupled, quasi-static, thermoelastic problem the basic equations are the

(*a*) heat conduction equation

$$kT_{,ii} = \rho c^{(v)}\dot{T} \quad \text{or} \quad k\nabla^2 T = \rho c^{(v)}\dot{T} \qquad (6.78)$$

(*b*) equilibrium equations

$$\sigma_{ij,j} + \rho b_i = 0 \quad \text{or} \quad \nabla \cdot \mathbf{\Sigma} + \rho \mathbf{b} = 0 \qquad (6.79)$$

(*c*) thermoelastic stress-strain equations

$$\sigma_{ij} = \lambda \delta_{ij}\epsilon_{kk} + 2\mu\epsilon_{ij} - (3\lambda + 2\mu)\alpha\delta_{ij}(T - T_0)$$

or $\qquad (6.80)$

$$\mathbf{\Sigma} = \lambda \mathbf{I}\epsilon + 2\mu \mathbf{E} - (3\lambda + 2\mu)\alpha \mathbf{I}(T - T_0)$$

(*d*) strain-displacement relations

$$\epsilon_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}) \quad \text{or} \quad \mathbf{E} = \tfrac{1}{2}(\nabla\mathbf{u} + \mathbf{u}\nabla) \qquad (6.81)$$

Solved Problems

HOOKE'S LAW. STRAIN ENERGY. ISOTROPY (Sec. 6.1-6.3)

6.1. Show that the strain energy density u^* for an isotropic Hookean solid may be expressed in terms of the strain tensor by $u^* = \lambda(\text{tr}\,\mathbf{E})^2/2 + \mu\mathbf{E}:\mathbf{E}$, and in terms of the stress tensor by $u^* = [(1+\nu)\mathbf{\Sigma}:\mathbf{\Sigma} - \nu(\text{tr}\,\mathbf{\Sigma})^2]/2E$.

Inserting (*6.21*) into (*6.13*), $u^* = (\lambda\delta_{ij}\epsilon_{kk} + 2\mu\epsilon_{ij})\epsilon_{ij}/2 = \lambda\epsilon_{ii}\epsilon_{jj}/2 + \mu\epsilon_{ij}\epsilon_{ij}$ which in symbolic notation is $u^* = \lambda(\text{tr}\,\mathbf{E})^2/2 + \mu\mathbf{E}:\mathbf{E}$.

Inserting (*6.24*) into (*6.13*), $u^* = \sigma_{ij}[(1+\nu)\sigma_{ij} - \nu\delta_{ij}\sigma_{kk}]/2E = [(1+\nu)\sigma_{ij}\sigma_{ij} - \nu\sigma_{ii}\sigma_{jj}]/2E$ which in symbolic notation is $u^* = [(1+\nu)\mathbf{\Sigma}:\mathbf{\Sigma} - \nu(\text{tr}\,\mathbf{\Sigma})^2]/2E$.

6.2. Separating the stress and strain tensors into their spherical and deviator components, express the strain energy density u^* as the sum of a dilatation energy density $u^*_{(S)}$ and distortion energy density $u^*_{(D)}$.

Inserting (*3.98*) and (*2.70*) into (*6.13*),

$$u^* = \tfrac{1}{2}(s_{ij} + \sigma_{kk}\delta_{ij}/3)(e_{ij} + \epsilon_{pp}\delta_{ij}/3) = \tfrac{1}{2}(s_{ij}e_{ij} + \sigma_{ii}e_{jj}/3 + s_{ii}\epsilon_{jj}/3 + \sigma_{ii}\epsilon_{jj}/3)$$

and since $e_{ii} = s_{ii} = 0$ this reduces to $u^* = u^*_{(S)} + u^*_{(D)} = \sigma_{ii}\epsilon_{jj}/6 + s_{ij}e_{ij}/2$.

6.3. Assuming a state of uniform compressive stress $\sigma_{ij} = -p\delta_{ij}$, develop the formulas for the bulk modulus (ratio of pressure to volume change) given in (*6.25*).

With $\sigma_{ij} = -p\delta_{ij}$, (*6.24*) becomes $\epsilon_{ij} = [(1+\nu)(-p\delta_{ij}) + \nu\delta_{ij}(3p)]/E$ and so $\epsilon_{ii} = [-3p(1+\nu) + 9p\nu]/E$. Thus $K = -p/\epsilon_{ii} = E/3(1-2\nu)$. Likewise from (*6.21*), $\sigma_{ii} = (3\lambda + 2\mu)\epsilon_{ii} = -3p$ so that $K = (3\lambda + 2\mu)/3$.

6.4. Express $u^*_{(S)}$ and $u^*_{(D)}$ of Problem 6.2 in terms of the engineering constants K and G and the strain components.

From a result in Problem 6.3, $\sigma_{ii} = 3K\epsilon_{ii}$ and so

$$u^*_{(S)} = \sigma_{ii}\epsilon_{jj}/6 = K\epsilon_{ii}\epsilon_{jj}/2 = K(\mathrm{I}_E)^2/2$$

From (*6.21*) and (*2.70*), $\sigma_{ij} = \lambda\delta_{ij}\epsilon_{kk} + 2\mu\epsilon_{ij} = s_{ij} + \sigma_{kk}\delta_{ij}/3$ and since $\sigma_{ii} = (3\lambda + 2\mu)\epsilon_{ii}$ it follows that $s_{ij} = 2\mu(\epsilon_{ij} - \epsilon_{kk}\delta_{ij}/3)$. Thus

$$u^*_{(D)} = 2\mu(\epsilon_{ij} - \epsilon_{kk}\delta_{ij}/3)(\epsilon_{ij} - \epsilon_{pp}\delta_{ij}/3)/2 = \mu(\epsilon_{ij}\epsilon_{ij} - \epsilon_{ii}\epsilon_{jj}/3)$$

Note that the dilatation energy density $u^*_{(S)}$ appears as a function of K only, whereas the distortion energy $u^*_{(D)}$ is in terms of μ (or G), the shear modulus.

6.5. In general, u^* may be expressed in the quadratic form $u^* = C^*_{KM}\epsilon_K\epsilon_M$ in which the C^*_{KM} are not necessarily symmetrical. Show that this equation may be written in the form of (*6.14*) and that $\partial u^*/\partial\epsilon_K = \sigma_K$.

Write the quadratic form as

$$u^* = \tfrac{1}{2}C^*_{KM}\epsilon_K\epsilon_M + \tfrac{1}{2}C^*_{KM}\epsilon_K\epsilon_M = \tfrac{1}{2}C^*_{KM}\epsilon_K\epsilon_M + \tfrac{1}{2}C^*_{PN}\epsilon_N\epsilon_P = \tfrac{1}{2}(C^*_{KM} + C^*_{MK})\epsilon_K\epsilon_M = \tfrac{1}{2}C_{KM}\epsilon_K\epsilon_M$$

where $C_{KM} = C_{MK}$.

Thus the derivative $\partial u^*/\partial\epsilon_R$ is now

$$\partial u^*/\partial\epsilon_R = \tfrac{1}{2}C_{KM}(\epsilon_{K,R}\epsilon_M + \epsilon_K\epsilon_{M,R}) = \tfrac{1}{2}C_{KM}(\delta_{KR}\epsilon_M + \epsilon_K\delta_{MR}) = \tfrac{1}{2}(C_{RM}\epsilon_M + C_{KR}\epsilon_K) = C_{RM}\epsilon_M = \sigma_R$$