Error analysis 2

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1. Elliptic case: (continued)

Recall we combined estimates for the error of polynomial interpolation with Cea's lemma:

continuity constant

coercivity constant

$$\|u-u_h\|_V \leq \frac{M}{\alpha}\|u-v_h\|_V \qquad \forall v_h \in V.$$

We used this to show convergence in 1d case:

If $u \in H^{p+1}(I)$, some $r \leq p$, then

$$\|u-u_h\|_V \leq \frac{M}{\alpha} C h^r |u|_{H^{r+1}(I)}$$

Higher dimensions: here the situation is more complicated because the accuracy of interpolation depends on the shape of each element:

<u>Thm</u>: if $r+1 \geq m$, and v_K^r the r^{th} degree interpolant of v on an element K, then

$$|v - v_K^r|_{H^m(K)} \leq C \frac{h_K^{r+1}}{\rho_K^m} |v|_{H^{r+1}(K)}$$

diameter of K, ie $\sup_{x,y\in K} |x-y|$

sphericity of K: diameter of largest sphere (ie circle in 2D) contained in K.

We say that a grid T_h is <u>regular</u> if $\exists\delta$:

h_K

$$\frac{h_K}{\rho_K} \leq \delta \qquad \forall K \in T_h$$

<u>Corollary</u>: if T_h is regular, and v_h^r is the r^{th} degree interpolant on T_h, then

$$|v - v_h^r|_{H^m(\Omega)} \leq C h^{r+1-m} |v|_{H^{r+1}(\Omega)}$$

<u>Corollary</u>: let u solve a variational problem

$$a(u,v) = F(v) \quad \forall v, \qquad \begin{cases} \text{continuous} & M \\ \text{coercivity} & \alpha \end{cases}$$

then if u_h^r Galerkin solution from degree r polynomials on a regular grid T_h, and $u \in H^{r+1}(\Omega)$, then

$$\boxed{\|u - u_h\|_{H^1(\Omega)} \leq \frac{M}{\alpha} C h^r |u|_{H^{r+1}(\Omega)}}$$

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2. Parabolic problems: here we consider equations in the weak form

$$\int_\Omega u_t v + a(u(t), v) = \int_\Omega f(t) v,$$

corresponding to equations $u_t + Lu = f(t)$, for an elliptic operator L.

a is assumed to be continuous and coercive, so we have existence/uniqueness.

given initial data $u_0 \in L^2(\Omega)$ and $f \in L^2(Q)$

cylinder $\{(x,t),\ t>0,\ x\in\Omega\}$

Restricting $u_h, v_h \in V_h$ with basis $\{\varphi_i\}$ results in the linear ODE

$$M u'(t) + A u(t) = f(t)$$

load

$\int \varphi_i \varphi_j$

stiffness, eg. for heat eqn mixed bcs.

$\int \nabla\varphi_i \cdot \nabla\varphi_j$

We will solve this ODE numerically. Outline of error analysis:

i) Stability of original equation

ii) Convergence of spatial discretization

iii) Convergence/stability of temporal discretization

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3. First, we note that $u_t u = \frac{1}{2}\frac{\partial}{\partial t}(|u|^2)$

$$\Rightarrow \int_\Omega u' u = \frac{1}{2}\frac{\partial}{\partial t}\|u\|^2_{L^2(\Omega)}$$

We then proceed by a familiar strategy: the equation holds for $v=u$, ie

$$\frac{1}{2}\frac{\partial}{\partial t}\|u\|^2_{L^2} + a(u,u) = \int_\Omega f u$$

using coercivity and Cauchy-Schwarz on the right, we find

$$\frac{1}{2}\frac{\partial}{\partial t}\|u\|^2_{L^2(\Omega)} + \alpha\|u\|^2_V \leq \|f\|_{L^2(\Omega)}\|u\|_{L^2(\Omega)}$$

can replace with
$\alpha|u|^2_{H^1} = \alpha\|\nabla u\|^2_{L^2(\Omega)}$ by Poincaré inequality

We then apply the inequality $ab \leq \varepsilon a^2 + \frac{b^2}{4\varepsilon}$ on the right: $(\forall\, \varepsilon > 0)$

$$\ldots \leq \frac{\tilde{C}^2}{2\alpha}\|f\|^2_{L^2} + \frac{\alpha}{2}|u|^2_{H^1}$$

(have used Poincaré again: $\|u\|^2_{L^2} \leq C|u|^2_{H^1}$)

Multiply equation by 2 and integrate in time:

$$\boxed{\|u\|^2_{L^2(\Omega)} + \alpha\int_0^t |u|^2_{H^1(\Omega)} \leq \|u_0\|^2_{L^2(\Omega)} + \frac{C^2}{\alpha}\int_0^t \|f(s)\|^2_{L^2(\Omega)}\,ds.}$$

this is our required a priori stability estimate (effectively in $\|u\|_{H^1}$-norm)

A similar estimate holds in L^2-norm.

Galerkin stability: an identical argument with u replaced by u_h in the above gives

$$\boxed{\|u_h\|^2_{L^2(\Omega)} + \alpha\int_0^t |u_h|^2_{H^1(\Omega)} \leq \|u_h(0)\|^2_{L^2(\Omega)} + \frac{C^2}{\alpha}\int_0^t \|f(s)\|^2_{L^2(\Omega)}\,ds}$$

typically $u_h(0)$ is an interpolation of $u(0)$ in the space V_h.

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4. For convergence, we follow as closely as we can the elliptic case: coercivity gives

$$\alpha \|u-u_h\|^2_{H^1} \leq \underbrace{a(u-u_h, u-v_h)}_{\leq M\|u-u_h\|_V \|u-v_h\| \text{ by continuity}} + \underbrace{a(u-u_h, v_h-u_h)}$$

No longer have Galerkin orthog., instead $= \left\langle \frac{\partial}{\partial t}(u-u_h), v_h-u_h \right\rangle_{L^2}$

i.e.

$$\alpha \|u-u_h\|^2_{H^1} \leq M\|u-u_h\|_V \|u-v_h\|_V - \left\langle \frac{\partial}{\partial t}(u-u_h), v_h-u_h \right\rangle$$

$$\leq \frac{\alpha}{2}\|u-u_h\|^2_V + \frac{M^2}{2\alpha}\|u-v_h\|_V + \left\langle \frac{\partial}{\partial t}(u-u_h), u-v_h \right\rangle$$

$$-\frac{1}{2}\frac{d}{dt}\|u-u_h\|^2_{L^2}$$

(as $\left\langle \frac{\partial}{\partial t}(u-u_h), u-u_h \right\rangle_{L^2} = \frac{1}{2}\frac{d}{dt}\|u-u_h\|^2_{L^2}$)

Multiplying by 2, collecting terms and integrating:

$$\|u-u_h\|^2_{L^2} + \alpha\int_0^t \|(u-u_h)(s)\|^2_{H^1}\,ds \leq \|(u-u_h)(0)\|^2_{L^2}$$

$$+\frac{M^2}{\alpha}\int_0^t \|(u-v_h)(s)\|^2_{H^1}\,ds + \underbrace{2\int_0^t \left\langle \frac{\partial}{\partial t}(u-u_h)(s), (u-v_h)(s) \right\rangle_{L^2} ds}.$$

integrate-by-parts:

$$\langle (u-u_h)(t), (u-v_h)(t)\rangle_{L^2} - \langle (u-u_h)(0), (u-v_h)(0)\rangle_{L^2}$$

$$- \int_0^t \langle (u-u_h)(s), \frac{\partial}{\partial t}(u-v_h)(s)\rangle_{L^2}\, ds.$$

$$\leq \frac{1}{4}\|u-u_h\|_{L^2}^2 + \|u-v_h\|_{L^2}^2 + \frac{1}{2}\|(u-u_h)(0)\|_{L^2}^2 + \frac{1}{2}\|(u-v_h)(0)\|_{L^2}^2$$

$$+ \int_0^t \|(u-u_h)(s)\|_{L^2}^2 \, \|\frac{\partial}{\partial t}(u-v_h)(s)\|_{L^2}^2\, ds.$$

Incorporating the blue text, RHS

$$\leq 2\|(u-u_h)(0)\|_{L^2}^2 + 2\int_0^t \|(u-u_h)(s)\|_{L^2}\, \|\frac{\partial}{\partial t}(u-v_h)(s)\|_{L^2}\, ds$$

$$+ \|u(0)-v_h(0)\|_{L^2}^2 + \|u-v_h\|_{L^2}^2 + \frac{M^2}{\alpha}\int_0^t \|(u-v_h)(s)\|_{H^1}^2\, ds.$$

Recall polynomial interpolation error:

$$h\|u-v_h\|_{H^1} + \|u-v_h\|_{L^2} \leq C h^{r+1} |u|_{H^{r+1}}$$

with the above, have

$$\leq C_1 h^{2r} N(u, u_t) + C_2 h^r \int_0^t |u_t(s)|_{H^r} \|(u-u_h)(s)\|_{L^2}\, ds.$$

And Gronwalling* gives

$$\boxed{\|u-u_h\|_{L^2}^2 + 2\alpha\int_0^t \|u-u_h\|_{H^1}^2 \leq C h^{2r}\left(\sqrt{N} + \frac{1}{2}\int_0^t |u_t|_{H^r}\right)^2}$$

*Gronwall's inequality: $(\varphi(t))^2 \leq g + \int_0^t A(s)\varphi(s)\, ds$

$$\Rightarrow \varphi(t) \leq \sqrt{g} + \int_0^t A(s)\, ds$$

(name used for

several related inequalities, here we have applied the usual form to $\log \varphi$; this form is given in Quarteroni, p. 28, Lemma 2.2).

Next: convergence and stability of the methods

$$M \frac{u^{k+1} - u^k}{\Delta t} + A\left(\vartheta u^{k+1} + (1-\vartheta) u^k\right) = \vartheta f^{k+1} + (1-\vartheta) f^k$$