Error analysis 1

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1. So far, we have only given a heuristic justification for our methods.

+ proved existence/uniqueness.

Want to show in addition:

i) Stability
ii) Convergence
iii) Error estimates & convergence

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2. Stability: we begin by noting that the Lax-Milgram theorem gives a stability result for weak solutions:

→ consider the problem

find $u \in V$: $a(u,v) = F(v) \quad \forall v \in V$

Hilbert space

continuous, coercive bilinear form

continuous, linear functional.

coercivity constant α, ie $a(u,u) \geq \alpha \|u\|_V^2$

Lax-Milgram ⇒ ∃ unique solution u

Moreover, the equation holds for $v = u$ as $u \in V$.

hence $F(u) \overset{①}{=} a(u,u) \geq \alpha \|u\|_V^2$ (by coercivity)

Now $|F(u)| \overset{②}{\leq} \|F\|_{V'} \|u\|_V$

where $\|F\|_{V'} = \sup_{v \in V} \dfrac{|F(v)|}{\|v\|_V}$

$v \neq 0$ $\vee$

in the dual norm

combining ② with the bound ① gives

$$\boxed{\|u\|_V \leq \frac{1}{\alpha}\|F\|_{V'}}$$

(NB, any continuous linear functional is necessarily bounded, ie RHS is finite)

→ Important: can set up the Galerkin method as a Lax-Milgram problem: if $V_h \subset V$ is a closed subspace, it is in itself a Hilbert subspace, and a is coercive/(continuous) with same coercivity coefficient

⇒ unique solution u_h satisfies:

$$\|u_h\|_V \leq \frac{1}{\alpha}\|F\|_{V_h'} \leq \frac{1}{\alpha}\|F\|_{V'}$$

$< \infty$.

i.e. the norm of approximate solution is bounded uniformly with respect to h.

→ also, guarantees that 'nearby' solutions (e.g. for different loads f_i) remain close

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3. Convergence: two ingredients

i) Galerkin orthogonality: let u_h be solution of Galerkin problem. Then

$$a(u-u_h, v_h) = 0 \qquad \forall v_h \in V_h$$

(Point: $a(u-u_h, v_h) = a(u, v_h) - a(u_h, v_h)$
$= F(v_h) - F(v_h) = 0$).

→ 'orthogonality' because if a is symmetric, $a(\cdot,\cdot)$ defines an inner product on V

⇒ u_h is the orthogonal projection on V_h of u

(i.e. minimizes distance $\|u-u_h\|_a$, where $\|\cdot\|_a$ is the 'energy norm' from inner product $a(\cdot,\cdot)$, i.e. $\|u\|_a^2 = a(u,u)$).

(NB: this interpretation is not necessary, and is only valid when $a(\cdot,\cdot)$ is symmetric)

ii) Céa lemma:

continuity constant of $a(\cdot,\cdot)$

$$\boxed{\|u-u_h\|_V \leq \frac{M}{\alpha} \|u-v_h\|_V \quad \forall v_h \in V_h}$$

coercivity constant of $a(\cdot,\cdot)$

Proof: Start with

$$\|u-u_h\|_V^2 \leq \frac{1}{\alpha} a(u-u_h, u-u_h)$$

(by coercivity of a)

$$\leq \frac{1}{\alpha}\left[a(u-u_h, u-v_h) + \cancel{a(u-u_h, v_h-u_h)}\right]$$

0 by Galerkin orthogonality

$$\leq \frac{M}{\alpha}\|u-u_h\|_V \|u-v_h\|_V$$

by continuity of a

we can cancel $\|u-u_h\|_V$ from both sides to obtain the result.

In particular, if V_h '$\longrightarrow$' V as $h \to 0$ we will obtain convergence. More precisely, need

$$\lim_{h\to 0} \inf_{v_h \in V_h} \|v - v_h\|_V = 0 \quad \forall v \in V$$

then find convergence of Galerkin method

i.e. $\lim_{h \to 0} \|u-u_h\|_V = 0$.

(NB conclusion of Céa can be strengthened for symmetric a: can replace constant by $\sqrt{M/\alpha}$)

→ in general, we will apply Céa's lemma together with theorems on polynomial interpolation error to obtain convergence (and rates of convergence too)

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4. First case: polynomial interpolation in 1D and 1D convergence rates.

the relevant theorem is: (we omit proof)

<u>Thm</u>: Suppose $v \in H^{r+1}(I)$ and v_h^r its degree r interpolation on a triangulation T_h

(with max element size h). Then

$$|v - v_h^r|_{H^k(I)} \lesssim \underbrace{C_{k,r}}_{\text{constant independent of } v,\, h.} h^{r+1-k} |v|_{H^{r+1}(I)}$$

$k = 0, 1$

(i.e. covers L^2 norm and H^1 seminorm)

<u>Corollary</u>: (combine with Céa's lemma)

Let u solve $\quad a(u,v) = F(v) \quad \forall v \in V$

for a continuous, coercive a with constants M, α respectively, and continuous F.

Let u_h be the approximate solution from the Galerkin scheme associated to piecewise polynomials of degree r on the elements $T_h \subseteq I$.

If $u \in H^{p+1}(I)$, some $r \leq p$. Then

$$\boxed{\|u - u_h\|_V \lesssim \frac{M}{\alpha} C h^r |u|_{H^{r+1}(I)}}$$

(i.e. have convergence of degree r in V-norm, typically H^1, provided u is in H^{r+1})

↳ i.e. increasing degree of interpolation increases degree of convergence as long as u regular enough.

5. Higher dimensions: here the situation is more complicated because the accuracy of interpolation depends on the shape of each element:

<u>Thm</u>: if $r+1 \geq m$, and v_K^r the r^{th} degree interpolant of v on an element K, then

$$|v - v_K^r|_{H^m(K)} \leq C \frac{h_K^{r+1}}{\rho_K^m} |v|_{H^{r+1}(K)}$$

diameter of K, ie $\sup_{x,y\in K} |x-y|$

sphericity of K: diameter of largest sphere (ie circle in 2D) contained in K.

We say that a grid T_h is <u>regular</u> if $\exists \delta$:

$$\frac{h_K}{\rho_K} \leq \delta \quad \forall K \in T_h$$

<u>Corollary</u>: if T_h is regular, and v_h^r is the r^{th} degree interpolant on T_h, then

$$|v - v_h^r|_{H^m(\Omega)} \leq C h^{r+1-m} |v|_{H^{r+1}(\Omega)}$$

<u>Corollary</u>: let u solve a variational problem

$a(u, v) = F(v)$ ... continuous M

Strategy: let u solve a variational problem

$a(u,v) = F(v) \quad \forall v,$ $\begin{cases} \text{continuous} & M \\ \text{coercivity} & \alpha \end{cases}$

then if u_h^r Galerkin solution from degree r polynomials on a regular grid T_h, and $u \in H^{r+1}(\Omega)$, then

$$\boxed{\| u - u_h \|_{H^1(\Omega)} \leq \frac{M}{\alpha} C h^r |u|_{H^{r+1}(\Omega)}}$$