2D poisson 2

torsdag 7. september 2017 13.29

1. Recall Galerkin problem:

$$Au = f$$

where $A_{ij} = \int_\Omega \nabla\varphi_i \cdot \nabla\varphi_j$, $f_j = \int_\Omega f\varphi_i$

corresponds to homogeneous Dirichlet problem

$$\begin{cases} -\nabla^2 u = f & \text{on } \Omega \qquad \Omega \subset \mathbb{R}^2 \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

→ Suppose we have a triangulation of the domain Ω into N triangles, and a function space V_h consisting of degree p polynomials on each element

→ Our strategy is the same as 1d case: we will loop through the N elements, and add the contribution from each element to A and f in the form of a subblock sub-vector.

→ Use:
- transformation to reference element
- barycentric coordinates
- Gaussian quadrature

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2. Transformation to reference element:

recall

$$X(\hat{x}) = \underline{x}_1 \phi_1(\hat{x}) + \underline{x}_2 \phi_2(\hat{x}) + \underline{x}_3 \phi_3(\hat{x})$$

$$= \underline{x}_1 \hat{x} + \underline{x}_2 \hat{y} + \underline{x}_3 (1 - \hat{x} - \hat{y})$$

$$= (\underline{x}_1 - \underline{x}_3)\hat{x} + (\underline{x}_2 - \underline{x}_3)\hat{y} + \underline{x}_3$$

$$= \underbrace{\left(\underline{x}_1 - \underline{x}_3 \;\middle|\; \underline{x}_2 - \underline{x}_3 \right)}_{\text{Jacobian } J \text{ of transformation, a } 2\times 2 \text{ matrix}} \begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix} + \underline{x}_3$$

Now $$\frac{\partial \varphi}{\partial x_i} = \sum_j \frac{\partial \varphi}{\partial \hat{x}_j} \underbrace{\frac{\partial \hat{x}_j}{\partial x_i}}_{J^{-1}_{ij}}$$

We then have

$$\nabla_x \varphi_i(x) = \underbrace{\nabla_{\hat{x}} \hat{\varphi}_i(\hat{x}) \;\backslash\; J}_{\text{i.e. solution of linear system}}$$

$$J y = \underbrace{\nabla_{\hat{x}} \hat{\varphi}_i(\hat{x})}$$

can compute using

can ✓ compute using reference elements, e.g.

<u>linear case:</u>

$$\nabla_{\hat{x}} \hat{\varphi}_i(\hat{x}) = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}$$

i.e. the i^{th} column of this matrix is $\nabla_{\hat{x}} \hat{\varphi}_i(\hat{x})$.

<u>Quadratic case:</u>

compute $\nabla_\lambda \hat{\varphi}_i(\lambda)$ in terms of barycentric coords. λ

$$\nabla_{\hat{x}} \hat{\varphi}_i(\hat{x}) = \underbrace{\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}}_{\text{Jacobian of mapping } \lambda \mapsto \hat{x}} \begin{pmatrix} 4\lambda_1 - 1 & 0 & 0 & 0 & 4\lambda_3 & 4\lambda_2 \\ 0 & 4\lambda_2 - 1 & 0 & 4\lambda_3 & 0 & 4\lambda_1 \\ 0 & 0 & 4\lambda_3 - 1 & 4\lambda_2 & 4\lambda_1 & 0 \end{pmatrix}$$

etc.

call P

Note: this computation 'vectorizes', i.e.

matrix $\nabla_x \varphi_i$ solves $\cancel{J \cdot \nabla_x \varphi_i}$ $(\nabla_x \varphi_i) \cdot J$ $= \left(\nabla_{\hat{x}} \hat{\varphi}_i(\hat{x})\right)^T$

call G.

equivalently, $J^T G^T = P$

i.e. $J \longrightarrow J^T$ is correction from old form.

then $(\nabla_x \varphi_i) \cdot (\nabla_x \varphi_j)$

$$= \sum_k G_{\cancel{ki}\,ik}\, G_{\cancel{kj}\,jk} = \cancel{G^T G}\ \ G G^T$$

the assignment.

Remember Jacobian!

$$\int_K (\nabla \varphi_i) \cdot (\nabla \varphi_j) = \int_{\hat{K}} \cancel{\phi} G(\hat{x})\, G(\hat{x})^T \, |J|$$

determinant

Evaluating integrals: in the linear case, the matrices $G(\hat{x})$ are constant on $\hat{K}$

$\Rightarrow$ the block contributing to A from element K is simply $\dfrac{\cancel{G^T} G\, G^T |J|}{2!}$,

$\frac{1}{2!}$ = the volume of $\hat{K}$, in higher dimensions this will become $d!$

where G solves $G\,[\underline{x}_1 - \underline{x}_3 \mid \underline{x}_2 - \underline{x}_3]\, \cancel{G} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}^T$

(in 3d, $[\underline{x}_1 - \underline{x}_4 \mid \underline{x}_2 - \underline{x}_4 \mid \underline{x}_3 - \underline{x}_4]\, G = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix}$ etc.)

(in 1d: $-h\, G = (1 \;\; -1)$, $|J| = h$, $\quad -h = x_1 - x_2$

so $G = \left(-\frac{1}{h}, \frac{1}{h}\right)$

and $\dfrac{G^T G\, |J|}{1!} = \dfrac{h}{h^2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = \dfrac{1}{h}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$,

as expected)

Higher elements: more complicated, because $G(\hat{x})$ is a function of $\hat{x}$ that differs for each element, and involves solving a linear system.

$\Rightarrow$ evaluate by Gaussian quadrature!

3. Gaussian quadrature: formulae

$$\int_K f(x)\,dx = \sum w_i f(x_i)\,|K|.$$

typically exact up to certain degree depending on choice of quadrature nodes / weights
(x_i) (w_i)

Usually give locations of nodes in barycentric coords,

e.g. $w_1 = 1$, $\lambda_1 = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ barycentre is exact at order 1 (i.e. on linear functions)

gives formula

$$\int_K f(x)\,dx \simeq |K| \cdot f(\lambda_1)$$

$$= |K| \, f\left(\tfrac{1}{3} \underline{x}_1 + \tfrac{1}{3} \underline{x}_2 + \tfrac{1}{3} \underline{x}_3 \right)$$

vertices of K.

→ tables of higher order quadrature rules can be found in Quarteroni 8.2.1 (p. 186)

4. Load vector: here we evaluate

$$\int_K f(x) \, \varphi_i(x)$$ by Gaussian quadrature

usually easiest to use barycentric coords on physical element,

ie $\simeq \sum_j w_i \, f(\lambda_j) \underbrace{\varphi_i(\lambda_j)}_{\text{known basis functions in terms of barycentric coords}} |K|$

remember to include volume of element! Use ... $|J|$ where element! Use $|K| = \frac{|J|}{2!}$, where

$|J|$ is Jacobian determinant

we computed when calculating

the stiffness matrix.