2D poisson 1

torsdag 7. september 2017 10.18

Consider
$$\begin{cases} -\nabla^2 u = f & \text{on } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

where $\Omega \subset \mathbb{R}^2$ is bounded by $\partial\Omega$.

1. Weak formulation:

$$\underbrace{-\int_\Omega (\nabla^2 u) v \, d\Omega}_{\text{'integrate-by-parts', i.e. use divergence theorem:}} = \int_\Omega f v \, d\Omega$$

'integrate-by-parts', i.e. use divergence theorem:

$$\int_\Omega \nabla \cdot a \, d\Omega = \int_{\partial\Omega} a \cdot n \, d\gamma$$

set $a = v\nabla u$ (use $\nabla\cdot(\varphi \underline{v}) = \varphi \nabla\cdot\underline{v} + \underline{v}\cdot\nabla\varphi$

$\Rightarrow \nabla\cdot(v\nabla u) = v\nabla^2 u + \nabla u \cdot \nabla v$)

$$\Rightarrow \int_\Omega v \nabla^2 u \, d\Omega = -\int_\Omega \nabla u \cdot \nabla v \, d\Omega + \underbrace{\int_{\partial\Omega} v \frac{\partial u}{\partial n}}_{0}$$

(Green's identity)

0 as v vanishes on boundary, in case of Dirichlet problem

i.e. Homogeneous dirichlet problem: find $u \in V$

$$\int_\Omega \nabla u \cdot \nabla v \, d\Omega = \int f v \, d\Omega \quad \forall v \in V$$

$$\boxed{\int_\Omega \nabla u \cdot \nabla v \, d\Omega = \int_\Omega f v \, d\Omega \quad \forall v \in V}$$

(take $V = H_0^1(\Omega)$)

Recap: square-integrable, with square-integrable distributional derivatives,

i.e. $\int_\Omega \frac{\partial u}{\partial x_i} \phi \, d\Omega = -\int_\Omega u \frac{\partial \phi}{\partial x_i} \quad \forall \phi$

'$v = 0$ on $\partial\Omega$' in sense that trace vanishes on boundary...

$\Rightarrow$ Galerkin problem: let $V_h \subset V$ be finite-dimensional, with basis $\{\varphi_i\}_{i \in I}$

$$\Rightarrow \int \nabla u \cdot \nabla v = \sum_{i,j} u_i v_j \underbrace{\int \nabla \varphi_i \cdot \nabla \varphi_j \, d\Omega}_{A_{ij}}$$

$$\int f \cdot v = \sum_j v_j \underbrace{\int_\Omega f \varphi_j \, d\Omega}_{f_j}$$

hence again, obtain linear system

$$\boxed{Au = f}$$

$\int_\Omega f \varphi_j \, d\Omega.$

$$\int_\Omega f \varphi_j \, d\Omega.$$

$$A_{ij} = \int_\Omega \nabla\varphi_i \cdot \nabla\varphi_j \, d\Omega$$

-> to appear in exercises: show existence & uniqueness of solution by continuity / coercivity etc.

e.g. => A is symmetric, +ve definite.

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2. Outlook for 2D Poisson:

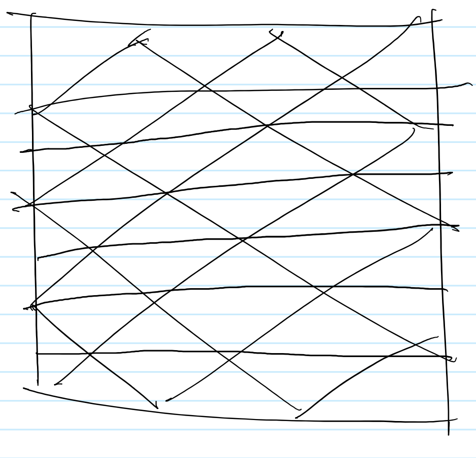
i) Weak solution (above, modifications for other boundary conditions)

ii) Geometry and choice of V_h, $\{\varphi_i\}$

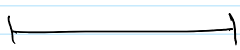
iii) Assembly: how to construct A_{ij} and f_j.

A major difference in ii/iii compared to 1D case is that elements can have different shape.

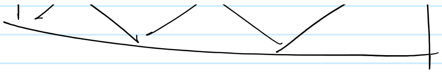
again, we will triangulate the domain ...



all 1d triangles are the same:



=> if linear elements used, 2x2 submatrix always $\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$



always $\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$

challenges: how to triangulate
(tackle much later)

how to construct φ_i, A_{ij} for different triangulations.

3. Lagrange basis functions & triangulations

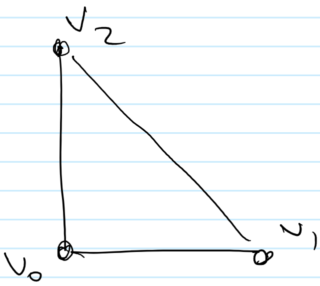
→ consider space of functions that are polynomials of a given degree on each element, ie

$$v(x,y) = \sum_{i+j \le d} a_{ij}^k \, x^i y^j \, , \quad x,y \in T_k$$

→ linear case: $v(x,y) = a_0 + a_1 x + a_2 y$

i.e. three 'degrees of freedom', a_0, a_1, a_2

take to be vertices:

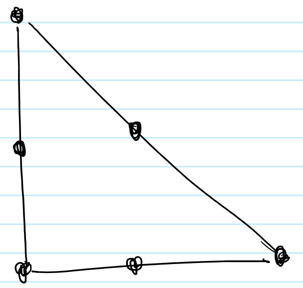


⇒ specifying values of v at vertices v_0, v_1, v_2 completely determines $v(x,y)$ on element.

→ quadratic case:

$$v(x,y) = a_0 + a_1 x + a_2 y + a_3 xy + a_4 x^2 + a_5 y^2.$$

i.e. six degrees of freedom:
take vertices and midpoints of edges



values at each of the six points determines $v(x,y)$ on element.

Challenge: determine the formula for a given φ_i on an element T, and find submatrix of A for each element.

Two helpful tools:

i) transformation to a 'reference element'

ii) use of 'barycentric coordinates'

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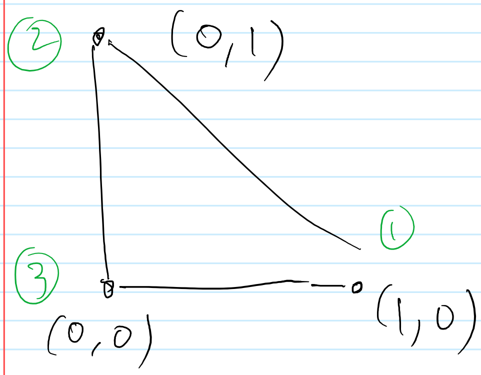
4. The reference element is

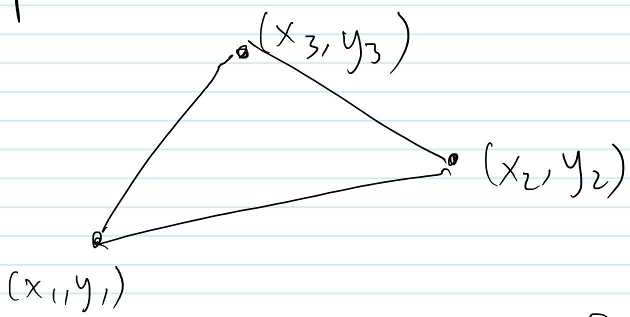


It is common to use hats to denote coords/basis functions on reference element, eg.

linear case:
$$\begin{cases} \hat{\varphi}_1 = x \\ \hat{\varphi}_2 = y \\ \hat{\varphi}_3 = 1 - x - y \end{cases}$$

= 1 at 1, 0 at 2, 3 etc.

Suppose we have an element



How to transform to reference element $\hat{K}$?

Use affine transform: $x \longmapsto \hat{x} = Kx + x_0$

$\longrightarrow$ to understand better, introduce barycentric coordinates

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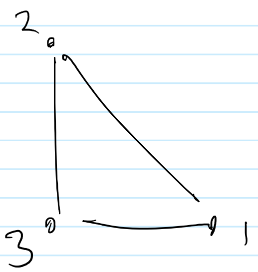
4b: The 'barycentric coordinates' on the simplex, $\varphi_i(\underline{x})$, coincide with the linear Lagrange basis functions, i.e.

$$\varphi_i(\underline{x}) = \begin{cases} 1 & \text{at } x_i \\ 0 & \text{at other vertices,} \end{cases}$$

and $\varphi_i(\underline{x})$ is a linear polynomial.

2d reference element:



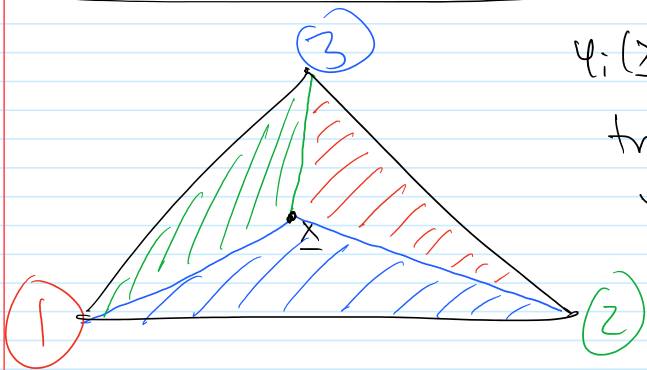
$$\hat{\varphi}_1 = \hat{x}$$
$$\hat{\varphi}_2 = \hat{y}$$
$$\hat{\varphi}_3 = 1 - \hat{x} - \hat{y}$$

(in n-d case, $\hat{\varphi}_i = \hat{x}_i$ for $i = 1, \dots, n-1$

$$\hat{\varphi}_n = 1 - \sum_{i=1}^{n-1} \hat{x}_i \quad)$$

These are coordinates in the sense that a given point $\begin{pmatrix} x \\ y \end{pmatrix}$ has coords $\begin{pmatrix} \varphi_1(x,y) \\ \varphi_2(x,y) \\ \varphi_3(x,y) \end{pmatrix}$

Geometric interpretation:



$\varphi_i(\underline{x})$ is ratio of area of triangle of same colour as vertex i to the whole triangle.

'barycentric' because the barycentre has coords $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ or $\frac{1}{d+1}$ in higher dimensions.

Mapping to reference element:

We can now map $\hat{x} \longmapsto x$ as follows:

$$\varphi_i(x) = \hat{\varphi}_i(\hat{x})$$

$$\underline{x}(\varphi(\underline{x})) = \underline{x}_1 \varphi_1(\underline{x}) + \underline{x}_2 \varphi_2(\underline{x}) + \underline{x}_3 \varphi_3(\underline{x})$$

$$\Rightarrow \quad x(\hat{x}) = \underline{x}_1 \hat{\varphi}_1(\hat{\underline{x}}) + \underline{x}_2 \hat{\varphi}_2(\hat{\underline{x}}) + \underline{x}_3 \hat{\varphi}_3(\hat{\underline{x}})$$

$$\Rightarrow \boxed{\underline{x}(\hat{\underline{x}}) = \underline{x}_1 \hat{\varphi}_1(\hat{\underline{x}}) + \underline{x}_2 \hat{\varphi}_2(\hat{\underline{x}}) + \underline{x}_3 \hat{\varphi}_3(\hat{\underline{x}})}$$

Note that substituting
$$\hat{\varphi}_1 = \hat{x}_1$$
$$\hat{\varphi}_2 = \hat{x}_2$$
$$\hat{\varphi}_3 = 1 - \hat{x}_1 - \hat{x}_2$$

gives the affine mapping if needed.

$\rightarrow$ in the sequel, we will apply this transform to start the assembly process. First...

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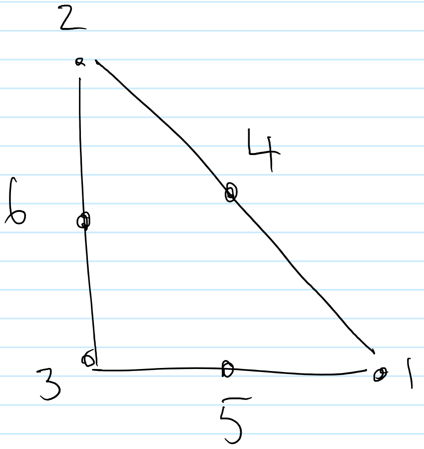
5. Quadratic elements in barycentric coords:

$\rightarrow$ <u>NB</u> here must distinguish basis functions φ_i and barycentric coords λ_i

$$\hat{\varphi}_i = \lambda_i(2\lambda_i - 1) \qquad i \text{ vertex}$$

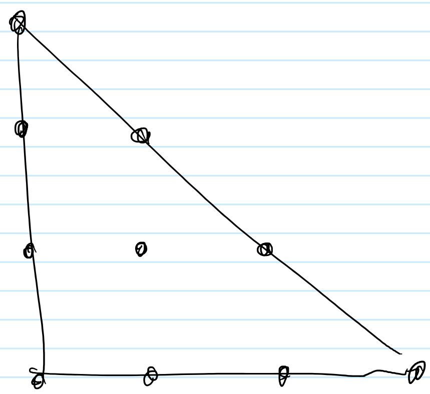
$$\varphi_k = 4\lambda_i\lambda_j \qquad \text{where } k \text{ is midpoint of edge joining } i, j$$



$$\varphi_1 = \lambda_1(2\lambda_1 - 1)$$
$$\vdots$$
$$\left.\begin{aligned} \varphi_4 &= 4\lambda_1\lambda_2 \\ \varphi_5 &= 4\lambda_1\lambda_3 \\ \varphi_6 &= 4\lambda_2\lambda_3 \end{aligned}\right\} \text{edges}$$

3 ... 5 $\qquad \varphi_6 = 4\lambda_2\lambda_3$)

→ Exercise: Cubic elements have 10 degrees of freedom:



How to write basis functions using barycentric coords?