Conclusion

torsdag 16. november 2017 11:24

1. One comment, relevant mostly for theoretical study of PDEs, is that if a PDE has a weak form

$$\text{find } u: \quad a(u,v) = F(v) \qquad \forall v \in V$$

and a is symmetric, it is equivalent to the minimization problem

$$u = \operatorname*{argmin}_{u \in V} \left\{ \tfrac{1}{2} a(u,u) - F(u) \right\}$$

→ eg, a possible weak form for the Poisson problem is

$$u = \operatorname*{argmin}_{u \in H^1(\Omega)} \left\{ \tfrac{1}{2} \int_\Omega |\nabla u|^2 - \int_\Omega f u . \right\}$$

→ It is possible to construct numerical methods based on optimization techniques but the end result will be something, like applying conjugate gradient to the minimization form of the linear system...
(i.e. we don't learn anything new)

2. <u>Continuation 1:</u> hyperbolic problems

the prototypical example is the linear transport-reaction equation (scalar case):

$$\left\{ \frac{\partial u}{\partial t} + b \cdot \nabla u + \sigma u = 0 \right.$$

$$\begin{cases} \dfrac{\partial u}{\partial t} + b \cdot \nabla u + \sigma u = 0 \\ u = \varphi \quad \text{on } \partial\Omega^{in}, \qquad u|_{t=0} = u_0 \text{ on } \Omega. \end{cases}$$

inflow boundary:

$x \in \partial\Omega$: $b \cdot n < 0$

(we assume $b = b(x)$ so $\partial\Omega^{in}$ constant in time, although this is not necessary).

We begin by discretizing in the familiar manner:

$$\int_\Omega \frac{\partial u_h}{\partial t} v_h + \int_\Omega v_h \, b \cdot \nabla u_h + \int_\Omega \sigma u_h v_h = \int_\Omega f v_h$$

where $u_h = u_h(t) \in V^h$

$v(t) \in V_h^{in}$ (ie V_h : $v_h|_{\partial\Omega^{in}} = 0$)

$\Rightarrow$ with appropriate basis functions, an equation of the form

$$M\dot{u} + Au = f, \text{ as before.}$$

a) discretizations: typically, the stability properties are somewhat better than for parabolic equations, but still must be careful

$\rightarrow$ stabilization can improve

b) <u>stabilization</u>: many of the same considerations (eg. artificial diffusion, least-squares regularization) arising from the diffusion-transport-reaction problem apply, with a bit of care!

3. <u>Continuation II</u>: Broader perspectives:

a) <u>Discontinuous Galerkin methods</u>:

One common strategy is to use finite elements, but not require continuity, i.e. each node has a different value for each element it intersects.

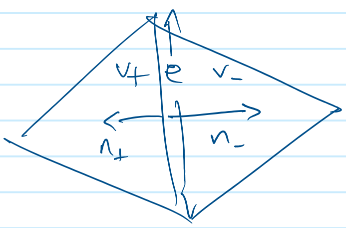
eg. a weak formulation for the Poisson equation in 2D is

$$\sum_{K_i} (\nabla u, \nabla v)_{K_i} + \sum_{e_i} \int_{e_i} \left([v]\{\nabla u\} + [\nabla u]\{v\} \right)$$

$$= \sum_{K_i} (f, v)_{K_i}$$

where $[v] = \frac{v_+ + v_-}{2}$, $\{v\} = \frac{v_+ n_+ + v_- n_-}{2}$ etc.

and e are edges, eg



etc.

-> typically stabilization terms are used

-> should be careful with boundary conditions.

-> find error bounds such as (for $r \geq s$)

$$\|u - u_h\|_E \lesssim C \cdot \left(\frac{h}{r}\right)^s r^{1/2} |u|_{H^{s+1}(\Omega)}$$

(ie exponential convergence in r for sufficiently smooth u).

-> commonly applied to transport problems, with some stabilization.

b) Petrov-Galerkin methods. :

Here we consider weak formulations of the type

find $u_h \in W_h$: $a(u_h, v_h) = F(v_h)$ $\forall v_h \in V_h$

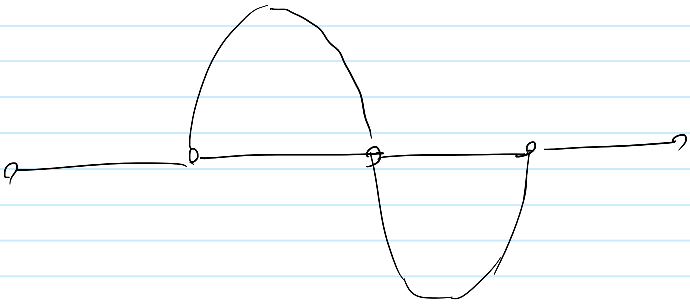
ie, the test functions and solution are in different spaces.

eg. can implement numerical viscosity for diffusion-transport problems in the following sense: (1d case)

let $\varphi_i(x)$ be standard linear basis functions for W_h.

construct $\psi_i(x) = \varphi_i(x) + \alpha B_i(x)$ (as basis for V_h)

where $B_i(x)$ are bubble functions



(quadratic interpolant, normalized to $\frac{3}{4}$ at midpoint).

then $\alpha = 1$ is upwind method

$\alpha = \coth(P) - \frac{1}{P}$ is Scharfetter-Gummel.

Need to be more careful with theoretical analysis though.

c) <u>Taylor-Galerkin</u>: we do not always treat time-dependent problems by first discretizing in space and then time independently.

e.g. consider 1d transport problem

$$\begin{cases} \dfrac{\partial u}{\partial t} + b\dfrac{\partial u}{\partial x} = 0 \\ u(x,0) = u_0(x). \end{cases}$$

then
$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial t}\left(-b\frac{\partial u}{\partial x}\right) = -b\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t}\right)$$
$$= b^2\frac{\partial^2 u}{\partial x^2}$$

If we then write the Taylor expansion + the remainder:

If we then write the Taylor expansion with remainder:

$$u(x,t^{n+1}) - u(x,t^n) = \Delta t \underbrace{\frac{\partial u}{\partial t}(x,t^n)}_{-b\frac{\partial u}{\partial x}} + \int_{t^n}^{t^{n+1}} (s-t^n)\, \underbrace{\frac{\partial^2 u}{\partial t^2}(x,s)}_{= b^2 \frac{\partial^2 u}{\partial x^2}} \, ds$$

If we then discretize the $\frac{\partial u}{\partial x}$ and $\frac{\partial^2 u}{\partial x^2}$ terms as normal, we get a scheme by evaluating the RHS integral by quadrature

eg, taking two quadrature nodes at $t^n + \frac{1}{3}\Delta t$, $t^n + \frac{2}{3}\Delta t$ gives

$$\left(M + \frac{b^2 (\Delta t)^2}{6} A \right) u^{n+1} = u^n$$

with fairly good stability and accuracy.

d) Spectral methods: here we consider the standard Galerkin framework, but where the basis elements of the finite dim function space will be orthogonal polynomials of some variety (including Fourier type)

→ typically very fast convergence for problems with smooth solutions.

→ difficult to handle complicated geometries.

→ non-sparse matrices with high condition numbers.

→ can be combined with finite element ideas (spectral elements). This resembles FEM with p-refinement

e) <u>p- and hp- FEM</u>: FEM where the adaptivity is in either polynomial degree alone (p-) or grid size and poly degree (hp-).

hp- methods are more complicated to implement and analyse, but can be very effective.