Convection 3

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1. We continue our study of artificial diffusion methods for the 1D transport-diffusion equation. We will prove the result

$$\|u-u_h\|_{H^1(\Omega)} \leq C \frac{h^r}{\mu(1+\phi(P))} \|u\|_{H^{r+1}(\Omega)}$$

$$+ \frac{\phi(P)}{1+\phi(P)} \|u\|_{H^1(\Omega)}$$

We begin by noting that the artificial diffusion is a generalized Galerkin scheme with $a_h(u,v) = \left(1+\phi(P)\right) a(u,v)$

Strang's lemma then gives

$$\|u-u_h\|_{H^1(\Omega)} \leq \inf_{w_h \in V_h} \left\{ \left(1+\frac{M}{\mu_h}\right) \|u-w_h\|_{H^1(\Omega)} + \frac{1}{\mu_h} \sup_{\substack{v_h \in V_h \\ v_h \neq 0}} \frac{\phi(P)\,|a(w_h,v_h)|}{\|v_h\|_{H^1(\Omega)}} \right\}$$

we let w_h be orthogonal projection on $H^1_0(\Omega)$ with respect to scalar product

$$\langle u,v\rangle_{H^1} = \int_0^1 u'v'$$

so that

$$\| \quad \| \qquad {}^{r} \|u\|_{H^{r+1}(\Omega)}$$

so that

$$\|w_h - u\|_{H^1(\Omega)} \leq C h^r \|u\|_{H^{r+1}(\Omega)}$$

and moreover

$$\frac{\phi(P)}{\mu_h} \frac{|a(w_h, v_h)|}{\|v_h\|_{H^1(\Omega)}} \leq \frac{\mu \phi(P)}{\mu_h} \|u\|_{H^1(\Omega)}$$

using $a(w_h, v_h) = \mu \int_0^1 w_h' v_h'$ and applying Cauchy-Schwarz

with the identity $\|w_h'\|_{L^2} \leq \|u'\|_{L^2}$

as $\dfrac{\mu \phi(P)}{\mu_h} = \dfrac{\phi(P)}{1 + \phi(P)}$, we obtain the result.

<u>Corollary</u>: in general, $\phi(P)$ depends on h

have $\phi(P) = \mathcal{O}(h)$ for upwind

$\phi(P) = \mathcal{O}(h^2)$ for Scharfetter-Gummel method.

(with fixed μ).

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2) 2D [illegible] diffusion: Consider a

2. <u>2D artificial diffusion</u>: Consider a problem of the form

$$\begin{cases} -\nabla\cdot(\mu\nabla u) + b\cdot\nabla u \;(+\;\sigma u) = f \\ u = 0 \quad \text{on} \quad \partial\Omega \end{cases}$$

the most immediate form of artificial diffusion is to add a term

$$Qh\int_\Omega \nabla u_h\cdot\nabla v_h \qquad (\text{i.e. } -Qh\nabla^2 u)$$

to the left hand side

('upwind artificial diffusion').

→ <u>Problem</u>: diffusion occurs crosswind as well as upwind!

In fact, want diffusion only in direction of $\underline{b}$, e.g. for 2D equation

$$-\mu\nabla^2 u + \frac{\partial u}{\partial x} = f \qquad \left(b = \begin{pmatrix}1\\0\end{pmatrix}\right)$$

would rather add $-Qh\dfrac{\partial^2 u}{\partial x^2}$ than

$$-Qh\nabla^2 u.$$

generally, can add ('Streamline diffusion')

$-Qh\, \nabla \cdot \left(b \dfrac{\partial u}{\partial b} \right)$, which gives a term

$$Qh \int_\Omega \frac{\partial u}{\partial b} \frac{\partial v}{\partial b}$$ in weak formulation.

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3. Analysis as a generalized Galerkin method:

suppose our method is to solve

$$a_h(u_h, v_h) = F_h(v_h) \quad \forall v_h \in V_h.$$

define truncation operator

$$\tau_h(u, v_h) = a_h(u, v_h) - F_h(v_h)$$

with norm

$$\tau_h(u) = \sup_{v_h \neq 0} \frac{|\tau_h(u, v_h)|}{\|v_h\|_V}$$

A generalized Galerkin scheme is

→ <u>consistent</u> if $\lim_{h \to 0} \tau_h(u) = 0$

→ <u>strongly consistent</u> if $\tau_h(u) = 0$.

<u>NB</u>: standard Galerkin is strongly consistent:

$$\tau_h(u, v_h) = a(u, v_h) - F(v_h) = 0$$

whilst generalized is typically consistent (but not strongly), using $(a_h - a) \longrightarrow 0$

$$(F_h - F) \longrightarrow 0$$

and Strang's lemma.

The upwind / streamline diffusion methods are consistent, but not strongly.

Will try to improve on $O(h)$ accuracy with stabilization using strongly consistent stabilizations.

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4. Galerkin least squares (GLS) stabilization.

Consider diffusion-transport-reaction equation
$$\begin{cases} Lu = f & \text{on } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$
(Divergence form)

with standard Galerkin form
$$a(u,v) = \underbrace{(f,v)}_{\text{i.e. } (f,v)_{L_2} = \int_\Omega fv}$$

we introduce the least squares stabilization

$$a(u_h, v_h) + \sum_K \tau_K (Lu_h, Lv_h)_{L^2} = (f,v)_{L^2}$$

$$+ \sum_K \tau_h (f, Lv_h)_{L^2}$$

or

$$\underbrace{a(u_h, v_h) + \sum_K \tau_K (Lu_h - f, Lv_h)_{L^2}}_{\text{call } L_h(u_h, f, v_h)} = (f, v)_{L^2}$$

for some stabilization parameter τ_K.

Now, as $L_h(u, f, v_h) = 0 \quad \forall v_h \in V$

(because $Lu - f = 0$)

the method is strongly consistent.

(NB other choices of stabilization L_h guaranteeing strong consistency exist).

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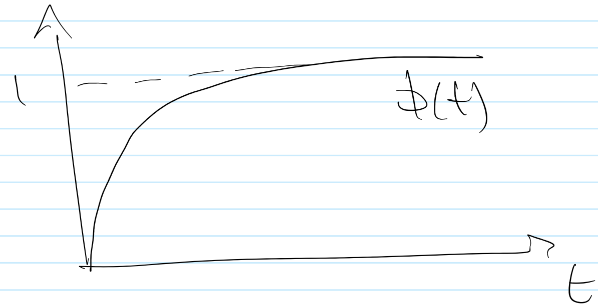
5. Some choices of stabilization τ_K:

a) $\tau_K = \delta \dfrac{h_K}{|b|}$, for some positive parameter δ.

b) $\tau_K = \dfrac{h_K}{2|b|} \phi(P)$, where $P = \dfrac{|b| h_K}{2\mu}$ is local Péclet number and ϕ some function, e.g.

$$\phi(t) = \coth(t) - \frac{1}{t}$$

(it is possible to recover Scharfetter-Gummel in this manner, for appropriate strongly consistent stabilization).



5. Convergence and stability of GLS:

write GLS in form

$$\bar{a}_h(u, v_h) = \bar{f}_h(v_h)$$

$$\bar{a}_h(u, v_h) = a(u_h, v_h) + \sum_K \delta\left(Lu_h, \frac{h_K}{|b|} Lv_h\right)_K.$$

$$\bar{f}_h(v_h) = (f, v_h) + \sum_K \delta\left(f, \frac{h_K}{|b|} Lv_h\right)_K.$$

Now define the norm

$$\|v_h\|^2_{GLS} = \mu \|\nabla v_h\|^2_{L^2} + \|\sqrt{\gamma}\, v_h\|^2_{L^2}$$
$$+ \sum_K \delta\left(\frac{h_K}{|b|} Lv_h, Lv_h\right)_K$$

where $\gamma = \frac{1}{2}\nabla\cdot b + \sigma$

(consider problem in divergence form, would replace with $-\frac{1}{2}\nabla\cdot b + \sigma$ in standard case).

i.e. defined so that $\|v_h\|^2_{GLS} = a_h^{(1)}(v_h, v_h)$

(identity analogous to our proof of coercivity)

We then have

Theorem:

a) Stability: let u_h solve GLS method.

For all $\delta > 0$, $\exists$ constant C independent of h:

$$\|u_h\|_{GLS} \leq C\|f\|_{L^2(\Omega)}.$$

b) Convergence: Assume V_h approximates V locally in the sense:

$$\|v - \hat{v}_h\|_{L^2(K)} + h_K|v - \hat{v}_h|_{H^1(K)} + h_K^2|v - \hat{v}_h|_{H^2(K)} \leq Ch_K^{r+1}|v|_{H^{r+1}(K)}$$

and that for each K, local Péclet no.

$$P_K(x) = \frac{|b(x)|\,h_K}{2\ldots} > 1 \quad \forall x \in K$$

$$P_K(x) = \frac{|b(x)|\, h_K}{2\mu} > 1 \quad \forall x \in K$$

suppose further we have the inverse inequality

$$\sum_K h_K^2 \int_K |\nabla^2 v_h|^2 \leq C_0 \|\nabla v_h\|_{L^2}^2$$

and that δ satisfies $0 < \delta \leq \frac{2}{C_0}$.

Then if $u \in H^{r+1}(\Omega)$, have

$$\boxed{\|u_h - u\|_{GLS} \leq C h^{r+\frac{1}{2}} |u|_{H^{r+1}(\Omega)}}$$

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6. Other strongly consistent methods:

an operator $L : V \longrightarrow V'$ is symmetric if

$$\langle u, Lv \rangle = \langle v, Lu \rangle$$

skew-symmetric if $\langle u, Lv \rangle = -\langle Lu, v \rangle$

then an operator can be split into sym and skew-sym parts, eg. let

$$Lu = -\mu \Delta u + \nabla \cdot (bu) + \sigma u$$

then

$$Lu = \left(\overbrace{-\mu \Delta u + \left(\sigma + \tfrac{1}{2} \nabla \cdot b\right) u}^{\text{sym}} \right).$$

$$Lu = \left(-\mu\Delta u + \left(c + \tfrac{1}{2}\nabla\cdot b\right)u\right)$$

$$+ \underbrace{\tfrac{1}{2}\left(\nabla\cdot bu + b\cdot\nabla u\right)}_{\text{skew-sym.}}$$

as $\nabla\cdot(bu) = \frac{1}{2}\nabla\cdot(bu) + \frac{1}{2}\nabla\cdot(bu)$

$$= \tfrac{1}{2}\nabla\cdot(bu) + \tfrac{1}{2}u\nabla\cdot b + \tfrac{1}{2}b\cdot\nabla u$$

can show

$$\tfrac{1}{2}\left(\nabla\cdot(bu), v\right) + \tfrac{1}{2}\left(b\cdot\nabla u, v\right)$$

$$= -\tfrac{1}{2}\left(u, b\cdot\nabla v\right) - \tfrac{1}{2}\left(u, \nabla\cdot(bv)\right).$$

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let $L = L_S + L_A$

L_S: sym

L_A: anti-sym (skew-)

for some parameter ρ, set stabilization

$$L_h(u_h, f, v_h) = \sum_K \left(Lu_h - f, (L_A + \rho L_S)v_h\right)$$

$\rho = 1$ gives GLS.

then $\rho = 0$ a (Streamline upwind Petrov-Galerkin)

$\rho = -1$ (Douglas - Wang)

are also reasonable choices of stabilization