Convection 2

torsdag 9. november 2017 12:32

We begin our study of methods to improve the performance of FEM for diffusion-transport-reaction equations

1. First we recall the 1D reaction-diffusion equation with constant coefficients:

$$\begin{cases} -\mu u'' + \sigma u = 0, & 0 < x < 1 \\ u(0) = 0, \quad u(1) = 1 \end{cases}$$

where $\mu, \sigma > 0$ constants

It was commented that the linear finite element method on a uniform grid generates oscillating solutions if

$$P = \frac{\sigma h^2}{6\mu} > 1$$

Contrast to the finite difference approx:

$$-\mu \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} + \sigma u_i = 0$$

$$u_0 = 0, \quad u_M = 1$$

This differs from linear finite elements in that the diagonal term $(\sigma I)u$ replaces the mass matrix $(\sigma M)u$.

It can be shown that the solution is

$$u_i = (\rho_+^M - \rho_-^M)^{-1} (\rho_+^i - \rho_-^i)$$

where $\rho_\pm = \frac{\gamma}{2} \pm \sqrt{\frac{\gamma^2}{4} - 1}$, $\gamma = 2 + \frac{\sigma h^2}{\mu}$.

which is monotone.

Practical implication: diagonalizing (lumping)

the mass matrix can in some cases help with stability of FEM

→ i.e., replace M with M_L, a diagonal matrix with entries

$$(M_L)_{ii} = \sum_j m_{ij} \quad \text{(i.e. row sums)}$$

[in 1D case, this coincides with finding M by trapezium method

$$\int_a^b f\,dx = \frac{b-a}{2}\left[f(a) + f(b)\right] \quad)$$

→ It can occur that M_L is singular.

Other diagonalization strategies exist, e.g. replace M with $\hat{M}$:

$$\hat{m}_{ii} = \frac{m_{ii}}{\sum_j m_{jj}}$$

[coincides in 1d case for linear/quadratic

(coincides in 1 d case for linear/quadratic elements)

nonsingular for Lagrange FEM...

2. Now consider 1D transport-diffusion equation

$$\begin{cases} -\mu u'' + bu' = 0, & 0 < x < 1 \\ u(0) = 0, \quad u(1) = 1 \end{cases}$$

where we recall linear finite elements / centred finite differences give a solution that oscillates if

$$P = \frac{|b|h}{2\mu} < 1$$

A simple method using finite differences to obtain a non-oscillating solution is to replace the central difference approx of u' with backward differences if $b > 0$

forward $b < 0$

– called 'upwinding', i.e. (assume $b > 0$)

$$-\mu \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} + b \frac{u_i - u_{i-1}}{h} = 0$$

$$u_0 = 0, \quad u_M = 1.$$

$\Rightarrow$ results in a scheme that converges

above: $O(h)$ rather than $O(h^2)$,
but gives monotone solution.

a) Artificial viscosity: the above strategy is specific to finite differences, but we can adapt to FEM, starting with the following observation:

$$\frac{u_i - u_{i-1}}{h} = \frac{u_{i+1} - u_{i-1}}{2h} - \frac{h}{2}\frac{u_{i+1} - 2u_i + u_{i-1}}{h^2}$$

$\Rightarrow$ the upwind scheme is equivalent to applying linear FEM / Centred FD on the equation

$$-\mu_h u'' + bu' = 0$$

where the artificial viscosity

$$\mu_h = \mu + \frac{bh}{2} = \mu(1+P)$$

the Péclet number of the perturbed equation is then

$$P^* = \frac{bh}{\mu(1+P)\,2} = \frac{P}{1+P} < 1$$

(i.e. the resulting scheme never oscillates).

b) Scharfetter–Gummel: can take other numerical viscosities than the upwind type:

in general, let $\mu_h = \mu(1 + \phi(P))$

where ϕ is some function

$\Rightarrow$ new Péclet number $P^* = \dfrac{P}{1 + \phi(P)}$

(eg $\phi(t) = t$ was upwind)

We obtain a second order scheme (instead of first order) by taking

$$\phi(t) = t - 1 + \frac{2t}{e^{2t} - 1}$$

$\rightarrow$ also, if f is piecewise constant, gives exact solution at nodes.

---

3. Stabilization in multidim case:

Here we consider artificial diffusion and related schemes. A possible framework is generalized Galerkin schemes:

discretize $a(u, v) = F(v) \quad \forall v$ by

$$a_h(u_h, v_h) = F_h(v_h) \quad \forall v_h \in V_h$$

where

$$a_h(u,v) = a(u,v) + b_h(u,v)$$

$$F_h(v) = F(v) + G_h(v).$$

artificial viscosity will contribute such a term.

i.e. $b_h(u,v) = \mu P(h)\, a(u,v)$

in 1d case

<u>Theorem</u>: artificial upwind viscosity method (1D case) has bound

$$\|u - u_h\|_{H^1(\Omega)} \leq C \frac{h^r}{\mu(1+\phi(P))} \|u\|_{H^{r+1}(\Omega)} + \frac{\phi(P)}{1+\phi(P)} \|u\|_{H^1(\Omega)}$$

For the above and related results, we will use

<u>Strang's lemma</u>: consider solving

$$a(u,v) = F(v) \qquad \forall v \in V$$

where a continuous, coercive, F linear, continuous st.

by generalized Galerkin method.

Assume

i) a_h is continuous and uniformly coercive, i.e. $\exists\, \alpha$ independent of h

$$: \quad a_h(v_h, v_h) \geq \alpha \|v_h\|_V^2 \qquad \forall h$$

ii) F_h is continuous, linear functional.

Then:

a) There exists a unique solution u_h to generalized Galerkin equation.

b) solution depends continuously on data:

$$\|u_h\|_V \leq \frac{1}{\alpha} \|F_h\|_{V_h'}$$

c) We have the error estimate

$$\|u - u_h\|_V \leq \inf_{w_h \in V_h} \left\{ \left(1 + \frac{M}{\alpha}\right) \|u - w_h\|_V + \frac{1}{\alpha} \|F - F_h\|_{V_h'} + \frac{1}{\alpha} \| a(w_h, \cdot) - a_h(w_h, \cdot)\|_{V_h'} \right\}$$

Results a+b are from Lax-Milgram applied to the discrete problem. For the remainder, we let $\sigma_h = u_h - w_h$, then

$$\alpha \|\sigma_h\|_V^2 \le a_h(\sigma_h, \sigma_h)$$

$$= a_h(u_h, \sigma_h) - a_h(w_h, \sigma_h)$$

$$= F_h(\sigma_h) - a_h(w_h, \sigma_h)$$

$$= \left[F_h(\sigma_h) - F(\sigma_h)\right] + F(\sigma_h) - a_h(w_h, \sigma_h)$$

$$= \left[F_h(\sigma_h) - F(\sigma_h)\right] + \underbrace{a(u, \sigma_h) - a_h(w_h, \sigma_h)}_{a(u - w_h, \sigma_h) + \left[a(w_h, \sigma_h) - a_h(w_h, \sigma_h)\right]}$$

if $\sigma_h \neq 0$, divide by $\alpha \|\sigma_h\|_V$

$\Rightarrow$

$$\|\sigma_h\|_V \le \frac{1}{\alpha}\left\{ \frac{|a(u - w_h, \sigma_h)|}{\|\sigma_h\|_V} + \frac{|F_h(\sigma_h) - F(\sigma_h)|}{\|\sigma_h\|_V} + \frac{a(w_h, \sigma_h) - a_h(w_h, \sigma_h)}{\|\sigma_h\|_V} \right\}$$

$$\le \frac{1}{\alpha}\left\{ M\|u - w_h\| + \|F_h - F\|_{V_h'} + \|a(w_h, \cdot) - a_h(w_h, \cdot)\|_{V_h'} \right\}$$

(inequality still valid for $\sigma_h = 0$, as RHS is true)

Now use $u - u_h = u - w_h - \sigma_h$, so

$$\|u-u_h\|_V \leq \|u-w_h\|_V + \|\sigma_h\|_V$$

$$\leq \left(1+\frac{M}{\alpha}\right)\|u-w_h\|_V + \frac{1}{\alpha}\|F_h - F\|_{V_h'}$$

$$+ \|a(w_h,\cdot) - a_h(w_h,\cdot)\|_{V_h'} \|$$

We are done, as can take inf on RHS.

NB: setting $w_h = u_h^*$, solution to ordinary Galerkin problem

$$\Rightarrow \|u-u_h\|_V \leq \|u-u_h^*\|_V + \frac{1}{\alpha}\|F-F_h\|_{V_h'}$$

$$+ \frac{1}{\alpha}\|a(u_h^*,\cdot) - a_h(u_h^*,\cdot)\|_{V_h'}$$

as the $\frac{M}{\alpha}\|u-u_h^*\|$ term disappears

when $a(u-u_h^*, \sigma_h) = 0$ by Galerkin orthogonality.

(i.e. shows error = Galerkin error + approx of a error.)