Convection

torsdag 2. november 2017 11:20

1. Here, we consider equations of the form

$$Lu = -\nabla\cdot(\mu\nabla u) + \underbrace{b\cdot\nabla u}_{\to\ \text{convection / advection / transport}} + \sigma u$$

reaction / absorption

$$\begin{cases} Lu = f & \text{on } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

where $\mu \in L^\infty$, $f, \sigma, \nabla\cdot b \in L^2$

a) <u>Weak formulation:</u>

multiplying by $v \in H^1$ and integrating gives

$$a(u,v) = (f,v)_{L^2}, \quad \text{where}$$

$$a(u,v) = \underbrace{\int_\Omega \mu\nabla u\cdot\nabla v}_{\text{as per usual analysis of } -\nabla\cdot(\mu\nabla u)} + \int_\Omega v\, b\cdot\nabla u + \int_\Omega \sigma u v$$

b) <u>Coercivity:</u> want $a(v,v) \geq a\|v\|_{H^1}^2$

A familiar analysis shows that

$$\int_\Omega \mu\nabla u\cdot\nabla v \geq C\cdot\mu_* \|v\|_{H^1}^2$$

$\to = \sup \mu(x)$

(exists as we assumed $\mu \in L^\infty$)

For the transport term, recall

$$\nabla \cdot (bu) = b \cdot \nabla u + u \nabla \cdot b$$

Now

$$\int_\Omega v b \cdot \nabla v \, d\Omega = \frac{1}{2} \int_\Omega b \cdot \nabla (v^2)$$

$$= \frac{1}{2} \int_\Omega \nabla \cdot (b v^2) - \frac{1}{2} \int v^2 \nabla \cdot b$$

$$= \frac{1}{2} \int_{\partial\Omega} \cancel{v^2} \frac{\partial b}{\partial n} - \frac{1}{2} \int v^2 \nabla \cdot b.$$

0 as $v = 0$ on $\partial\Omega$.

We can combine this with the reaction term, obtaining

$$\int_\Omega v b \cdot \nabla v + \int_\Omega \sigma v^2 = \underbrace{\int_\Omega v^2 \left(\sigma - \frac{1}{2} \nabla \cdot b \right)}_{\text{positive if } \sigma - \frac{1}{2} \nabla \cdot b \geq 0}.$$

eg. $\alpha(v, v) \geq \alpha \|v\|^2_{H^1} + \text{positive}$

$$\geq \alpha \|v\|^2_{H^1},$$

so certainly $\sigma - \frac{1}{2} \nabla \cdot b \geq 0$ in a sufficient condition for coercivity.

c) Continuity: simpler, eg

$$\left| \int_\Omega v\, b \cdot \nabla u \right| \leq \|b\|_{L^\infty} \|v\|_{L^2} \|\nabla u\|_{L^2}$$

$$\leq \|b\|_{L^\infty} \|v\|_{H^1} \|u\|_{H^1}$$

i.e. can show continuity
$$|a(u,v)| \leq M \|u\|_{H^1} \|v\|_{H^1}$$

where $M = \|\mu\|_{L^\infty} + \|b\|_{L^\infty} + \|\sigma\|_{L^2}$ etc.

2. Galerkin scheme: let $\{\varphi_i\}$ be a basis for $V_h \subset H^1$

$\Rightarrow$ equation becomes $Au = f$

where

$$A = \int_\Omega \mu \nabla \varphi_i \cdot \nabla \varphi_j + \underbrace{\int_\Omega \varphi_i\, b \cdot \nabla \varphi_j}_{\text{not symmetric!}} + \int_\Omega \sigma \varphi_i \varphi_j$$

Now, many of the previous results hold in this context, e.g.

$$\|u - u_h\|_V \leq \frac{M}{\alpha} \inf_{v_h \in V_h} \|u - v_h\|_V$$

However, note:

$$M = \|\mu\|_{L^\infty} + \|b\|_{L^\infty} + \|\sigma\|_{L^2}$$

whilst $\alpha \simeq \|\mu\|_{L^\infty}$

$\Rightarrow$ the constant becomes large if the transport / reaction terms dominate the diffusion

$\Rightarrow$ we will see that these situations can cause numerical problems.

3. Divergence form:

$$\begin{cases} Lu = \nabla \cdot (-\mu \nabla u + \underline{b} u) + \sigma u = f \\ u = 0. \end{cases}$$

with weak formulation

$$a(u,v) = \int_\Omega (\mu \nabla u - \underline{b} u) \cdot \nabla v + \int_\Omega \sigma u v$$

$$= f(v)$$

(the forms are equivalent for solenoidal b as $\nabla \cdot (bu) = b \cdot \nabla u + u \nabla \cdot b$)

It can be shown that a sufficient condition for coercivity is $\frac{1}{2} \nabla \cdot b + \sigma \geq 0$

4. Observations from the one-dimensional case:

At present, we have already justified that our basic Galerkin FEM scheme works in some sense.

Nonetheless, we will show that in some cases, the behaviour in practice is not satisfactory, and indicate some possible improvements.

We illustrate the potential issues by an examination of the 1D case

$$\begin{cases} -\mu u'' + bu' + \sigma u = 0 & 0 < x < 1 \\ u(0) = 0, \quad u(1) = 1 \end{cases}$$

μ, b, σ constant.

a) Diffusion - transport $(\sigma = 0)$:

The characteristic equation becomes

$$\lambda(b - \mu\lambda) = 0$$

$$\Rightarrow \quad \lambda = 0 \text{ or } \frac{b}{\mu}$$

hence, applying boundary conditions,

$$u(x) = \frac{\exp\left(\frac{b}{\mu}x\right) - 1}{\exp\left(\frac{b}{\mu}\right) - 1}$$

clearly, the behaviour is determined by the ratio $\frac{b}{\mu}$:

i) $\frac{b}{\mu}$ small (i.e., diffusion dominates transport.)

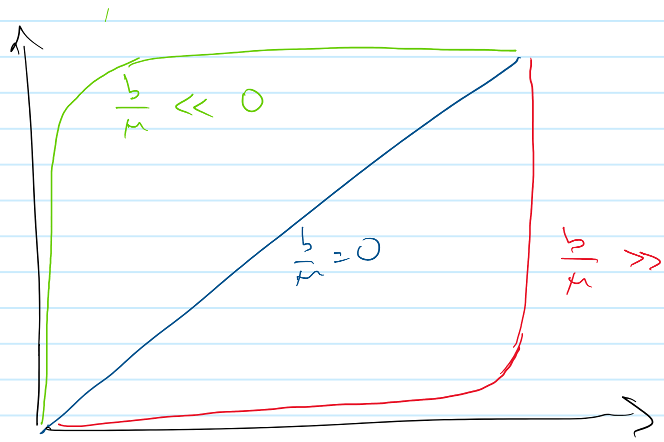
$$\exp\left(\frac{b}{\mu}x\right) - 1 \simeq \frac{b}{\mu}x.$$

$$\Rightarrow u(x) \simeq \frac{\frac{b}{\mu}x}{\frac{b}{\mu}} = x.$$

ii) $\frac{b}{\mu}$ large: (transport dominates diffusion)

$$u(x) \simeq \frac{\exp\left(\frac{b}{\mu}x\right)}{\exp\left(\frac{b}{\mu}\right)} = \exp\left(\frac{b}{\mu}(x-1)\right)$$

the different situations:



- <u>FEM solution</u>: this problem is of inhomogeneous Dirichlet type, so must construct a lifting:

here, let's lift by $R_g = x$

(instead of usual interpolation of boundary data)

$\Rightarrow \quad \mathring{u} = u - R_g = u - x$ solves homogeneous problem.

$$\underbrace{a(\mathring{u}, v)}_{\int \mu \varphi_i' \varphi_j' + b \varphi_i' \varphi_j} = -a(x, v) = \underbrace{-\int_0^1 b v \, dx}_{f(v)}.$$

$\int \mu \varphi_i' \varphi_j' + b \varphi_i' \varphi_j$ gives discretization A_h

i.e. the function $R_g(x) = x$.

- Suppose now we have a uniform grid, and take linear basis functions

$\Rightarrow$ we can show relations

$$\frac{\mu}{h}\left(-u_{i-1} + 2u_i - u_{i+1}\right) + \frac{b}{2}\left(u_{i+1} - u_{i-1}\right) = 0$$

(as from a finite difference scheme)

$\Rightarrow$ the recurrence

$$(P-1)\, u_{i+1} + 2u_i - (P+1)\, u_{i-1} = 0,$$

where $P = \dfrac{|b|\, h}{2\mu}$, local Péclet number

(or h with lifting $\Rightarrow R_g$ will be a particular solution)

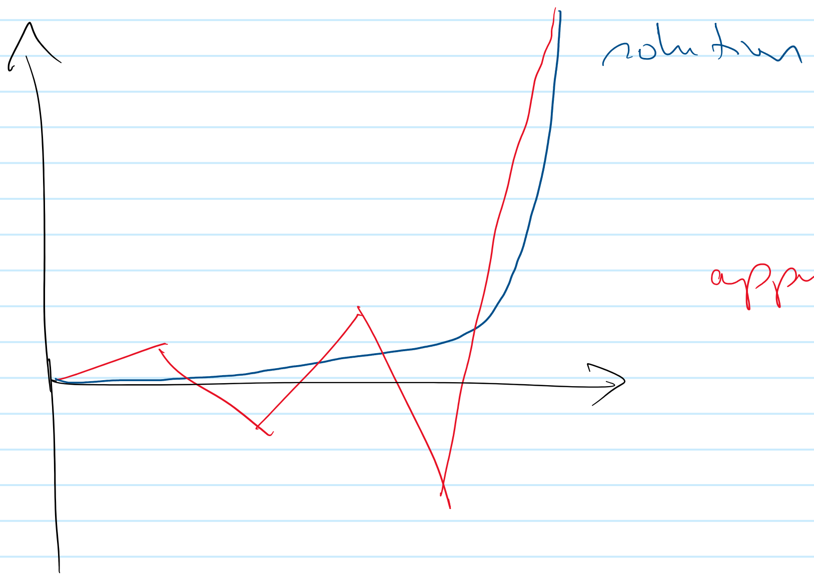
solving the associated characteristic equation gives (with appropriate ...

contribution from lift)

$$u_i = \frac{1 - \left(\frac{1+P}{1-P}\right)^i}{1 - \left(\frac{1+P}{1-P}\right)^M}$$

=> oscillatory solution if $P > 1$
(with increasingly large oscillations).

e.g.



for sufficiently small h, $P < 1$, but this may involve taking impractically small h.

Alternatives

-> adaptive grids (typically refines strongly near boundary).

-> smarter discretizations,
(e.g. increasing contribution of diffusion to help smooth - we will consider

this later)

b) Reaction-diffusion $(b=0)$

Here we observe similar phenomena

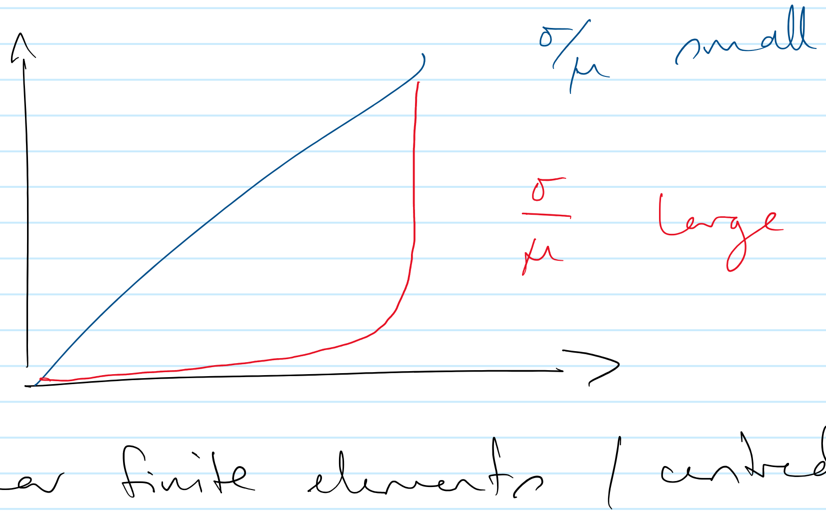
Indeed, the exact solution to

$$\begin{cases} -\mu u'' + \sigma u = 0 & 0<x<1 \\ u(0)=0, \quad u(1)=1 \end{cases}$$

is $u(x) = \dfrac{\sinh(\alpha x)}{\sinh(\alpha)}$, $\quad \alpha = \sqrt{\sigma/\mu}$

(assume μ and σ are both positive)

i.e. similar behaviour:



linear finite elements / centred finite differences result in recursions of the form

$$(P-1)\, u_{i+1} + 2(1+2P)\, u_i + (P-1)\, u_{i-1} = 0$$

where here the local Péclet number is

$$P = \frac{\sigma h^2}{6\mu}.$$

$\Rightarrow$ more complicated solution, but again oscillatory for $P > 1$

c) Similar considerations apply in the presence of transport & reaction together, but exact solutions are yet more complicated.