1d galerkin 2

onsdag 23. august 2017 13.20

We demonstrated the overall Galerkin cycle:

1. Weak solution, eg. $\int u'v' = \int F v \quad \forall v \in V$
   $v(0) = v(1) = 0$

2. Choice of function space V_h
   and basis φ_i

3. "Assembly": reduction to linear system
   $$A u = F$$
   must be constructed, eg. by quadrature

4. Boundary conditions: some rows etc. of A, F are deleted to comply.

5. Solve system for u.

Today we focus on different boundary conditions.

- note that this impacts both 1 and 4 in above cycle.
- as you will soon discover, this becomes harder in higher dimensions!

eg. $-u''(x) = f(x), \quad 0 < x < 1$

i) $u(0) = a, \quad u(1) = b$ (inhomogeneous Dirichlet)

ii) $u'(0) = a, \quad u'(1) = b$ (Neumann).

iii) $u(0) = a, \quad u'(1) = b$ (mixed)

iv) $u(0) = a, \quad u'(1) + \gamma u(1) = b$ (Robin)

γ given constant.

We first tackle i). Here, note that we used $-\int u''v = \int u'v'$, that u, v vanished at endpoints.

note, ~~eg~~ a space with $v(0) = a$, $v(1) = b$ is not a vector space.

For $u(0) = a$, $u(1) = b$ case, we relate to homogeneous case by linearity:

<u>idea:</u> if $\hat{u}$ solves homogeneous problem, then

$$u := \hat{u} - \underbrace{(1-x)a + xb}_{\text{'lifting' of boundary data}} \text{ solves.}$$

(for Poisson equation)

weak form:

In general, write $R_g = \varphi_0 a + \varphi_N b$.

basis functions at endpoints

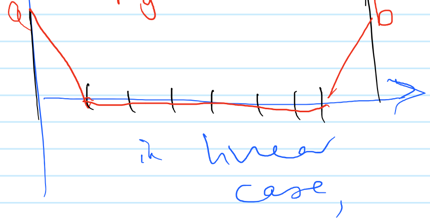
(lifting: more complicated in higher dimension)

then $u_h = \hat{u}_h + R_g$, and

$$\int u'v' = \int fv \quad \text{becomes}$$

$$\underbrace{\int \hat{u}'v' = \int fv - \int R_g' v'}_{\text{yields a system } A\hat{u} = \hat{F}},$$

R_g:



in linear case, etc.

solve for $\hat{u}$, then set $u = \hat{u} + R_g$.

here $\int_0^1 R_g' v' = \sum v_i \int_0^1 (a\varphi_0' + b\varphi_N')\varphi_i'$

$= \sum v_i \left(a \int_0^1 \varphi_0' \varphi_i' + b \int_0^1 \varphi_N' \varphi_i' \right)$

$= v^T (a\, A_1 + b A_N)$ ↳ last column of A

↳ first column of stiffness matrix A (before implementing boundary conditions)

$$\boxed{\begin{aligned} A\hat{u} &= F - aA_1 - bA_N \\ u &= \hat{u} + a\varphi_0 + b\varphi_N. \end{aligned}}$$

Now tackle ii): Neumann problem.

NB solution only determined up to a constant

here $u'(0) = a$, $u'(1) = b$.

Now $\int_0^1 u'' v = [u'v]_0^1 - \int u'v'$

$= bv(1) - av(0) - \int u'v'$

i.e. obtain

$$\int_0^1 u'v' = \int_0^1 fv + bv(1) - av(0) \quad \forall v \in V$$

$$\int_0^1 u'v' = \int_0^1 fv + bv(1) \qquad \forall v \in V$$

i.e. $Au = F + (-a, 0, \dots, 0, b)^T$

but we no longer remove the rows corresponding to φ_0, φ_N.

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iii) Mixed: $u(0) = 0, \quad u'(1) = b$

(homogeneous)

solution unique again!

$$\int_0^1 u''v = \underbrace{[u'v]_0^1}_{bv(1)} - \int_0^1 u'v'$$

$bv(1)$, enforce $v(0) = 0$

i.e. construct

$$Au = F + (0, \dots, 0, b)^T,$$

remove entries corresponding to φ_0, and solve.

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iv) Robin: $u(0) = 0, \quad u'(1) + \gamma u(1) = b$

(Optional)

Here $[u'v]_0^1 = u'(1)v(1)$

$$= (b - \gamma u(1))\, v(1)$$

and equation to solve becomes

$$(A + \gamma e_{NN})u = F + (0, \dots, 0, b)^T$$

matrix 1 in $(N,N)^{th}$ entry
0 elsewhere. etc.

with γ_0 entries removed.