Error estimates

torsdag 19. oktober 2017 09:37

1. Here we discuss briefly error estimates in the L^2 and L^∞-norms.

a) L^2 norm: first, recall the following identity:

$$C_1 h^{d/2} |v|_2 \leq \|v_h\|_{L^2} \leq C_2 h^{d/2} |v|_2$$

where $|v|_2$ is the 2-norm of the vector v (i.e. that stored on your machine!)

and $v_h = \sum_i v_i \varphi_i(x)$ is the associated function.

In practice: suppose we have an approx. solution v, which we wish to compare to the exact solution u. We can then estimate the error by

$$\|u-v\|_{L^2} \simeq h^{d/2} |u-v|_2.$$

For this, we need to know h!

Roughly speaking, we can approximate $h^d \sim N^{-1}$ where N is no. of nodes, and take

$$\|u-v\|_{L^2} \simeq N^{-1/2} |u-v|_2,$$

but this might not be completely accurate (e.g. in the case of the getSphere triangulation, h seems to scale like

$$h^{3.7} \sim N^{-1}$$).

b) L^∞-norm: Here error estimation is more complicated.

In general, we use the following error estimate for the error of polynomial interpolation:

$$\| v - v_K^r \|_{L^\infty(K)} \leq \frac{c}{|K|^{1/2}} h_K^{r+1} |v|_{H^{r+1}(K)} .$$

Now under a suitable regularity assumption, we have

$$|K| \leq c h^d ,$$

(1) hence

$$\| v - v_K^r \|_{L^\infty(K)} \leq c h^{r+1-\frac{d}{2}} |v|_{H^{r+1}(K)}$$

Moreover, it can be shown that

(2) $$\| u_h - u_K^k \|_{L^\infty(K)} \leq \underbrace{c |K|^{-1/2}}_{c h^{-d/2}} \| u_h - u_K^k \|_{L^2(K)}$$

where u_K^k is the interpolant of the exact solution

(this is based on equivalence of finite dimensional norms).

We then have

$$\|u - u_h\|_{L^\infty(\Omega)} \leq \underbrace{\|u - u_h^k\|_{L^\infty(\Omega)}}_{1} + \underbrace{\|u_h^k - u_h\|_{L^\infty(\Omega)}}_{2}$$

$$\boxed{\|u - u_h\|_{L^\infty(\Omega)} \leq C h^{r+1-\frac{d}{2}} |v|_{H^{r+1}(K)}}$$

(where we used a bound

$$\|u_h - u_K^k\|_{L^2(\Omega)} \leq C h^{r+1} |u|_{H^{r+1}(\Omega)},$$

coming from the L^2 error bound of u_h and the polynomial interpolation error in L^2:

$$\|v - v_h^k\|_{L^2} \leq C h^{r+1} |v|_{H^{r+1}(\Omega)}$$

).

Note that this bound is not always optimal. It's complicated!

---

2. Proof of L^2 convergence (Poisson case) :

$\downarrow$ homogeneous Dirichlet

Let $e_h = u - u_h$

consider equation

$$\begin{cases} -\nabla^2 \phi = e_h & \text{in } \Omega \end{cases}$$

$$\begin{cases} -\nabla^2 \phi = e_h & \text{in } \Omega \\ \phi = 0 & \text{on } \partial\Omega. \end{cases}$$

i.e. $\quad a(\phi, v) = \int_\Omega e_h v$

let $v = e_h \Rightarrow \quad a(\phi, e_h) = \|e_h\|^2_{L^2(\Omega)}$

thus

$a(\phi, e_h) = a(\phi - \phi_h, e_h)$,

where ϕ_h = orthogonal projection of ϕ in V_h

(i.e. polynomial interpolation)

then

$$\|e_h\|^2_{L^2(\Omega)} \leq |e_h|_{H^1(\Omega)} \, |\phi - \phi_h|_{H^1(\Omega)}$$

$$\leq |e_h|_{H^1(\Omega)} \cdot Ch|\phi|_{H^2(\Omega)}$$

It can be proven that

$$|\phi|_{H^2(\Omega)} \leq C\|e_h\|_{L^2(\Omega)}$$

(for Poisson equation, this is an elliptic regularity result)

Hence

$$\|e_h\|^2_{L^2(\Omega)} \leq Ch\|e_h\|_{H^1(\Omega)}$$

and we gained an order of convergence

(more generally, consider adjoint problems...)