Adaptivity

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1. Here we examine strategies by which the grid / basis functions can be generated or modified to achieve a desired accuracy with as little computational time possible.

Two strategies:

→ A priori adaptivity: here the estimates employed do not require us to solve the problem (or at least we do so with a very coarse approximation); the grid is constructed before solving any system

→ A posteriori adaptivity: our estimates make use of our solution, so we must remesh and solve again (possibly multiple times).

An important (but difficult to achieve!) principle: equidistribution of errors.

→ in an ideal world, the contribution to the global error from each element should be roughly the same

⇒ typically $h_K^r |u|_{H^{r+1}(K)} \approx$ constant for all K.

(recall

$$\|u-u_h\|_{H^1(\Omega)} \leq \frac{M}{\alpha} C h^r |u|_{H^{r+1}(\Omega)}.)$$

→ to increase accuracy, we can refine the grid or increase the polynomial degree p (or both!)

→ here we consider grid refinement only, but note that doing both can give very effective results ('hp-refinement').

Both a priori and a posteriori refinement rely on an estimation of $|u|_{H^{r+1}(K)}$.
In fact, our estimates use

$$\|u-u_h\|^2_{H^1(\Omega)} \leq \frac{M}{\alpha} C \sum_K h_K^{2r} |u|^2_{H^{r+1}(K)}.$$

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2. A priori adaptivity:

a) Here we will use a coarse approximation to u to estimate $|u|_{H^2(K)}$.

↓

assume we compute u_h on some fairly coarse grid T_h.

Assume we are working with linear finite elements. Then we have a problem, because the approximation $u_h \notin H^2(K)$.

$\Rightarrow$ use <u>reconstruction</u>.

The idea is to find a continuous, piecewise linear approximation of ∇u_h

(at the moment, ∇u_h is discontinuous, but constant on elements).

i.e. take $(g_h)_j(x) = \sum_i g_{ij}\, \varphi_i(x)$

a vector, $g_h \approx \nabla u_h$.

sum over nodes.

<u>Algorithm</u>: we write $g_{ij} = \Pi_{ij}(x_i)$

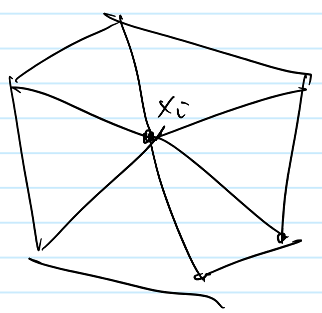
where $\Pi_{ij}(\underline{x}) = \underline{\underline{A}}\,\underline{x} + \underline{b}$ — the node x_i

(an affine map)

is defined by minimizing

$$\left\| \Pi_{ij} - \frac{\partial u_h}{\partial x_j} \right\|_{L^2(K_i)} \qquad j = 1, \ldots, d.$$

where K_i is the set of all elements containing the node x_i, eg.



K_i

<u>NB</u>: the integrals $\int_{K_i} \left(A_{ij} x + b_{ij} - \frac{\partial u_h}{\partial x_j} \right)^2$

NB: the integrals $\int_{K_i} \left(A_{ij} x + b_{ij} - \frac{\partial u_h}{\partial x_j} \right)^2$ can be computed analytically, giving a system to be solved; or quadrature results in an affine least-squares problem.

We then construct

$$H(u) \simeq \frac{1}{2}\left(\nabla g_h + \nabla g_h^T\right)$$

(to inforce symmetry of Hessian)

ie Hessian $H(u)_{ij} = \frac{\partial^2 u}{\partial x_i \partial x_j}$

b) refinement: we would like to have

$$\sum_K h_K^2 |u|^2_{H^2(K)} \simeq \sum h_K^2 \sum_{i,j} \|H_{ij}\|^2_{L^2} \leq \frac{\varepsilon^2}{C^2}$$

to obtain error ε.

we can refine by inserting new vertices inside elements for which

$$\sum_K h_K^2 |u|^2_{H^2(K)} > \frac{1}{N}\left(\frac{\varepsilon}{C}\right)^2$$

no. of elements.

c) Remeshing: here construct a spacing function

(constant on each element)

$$H| = \frac{\varepsilon}{C\sqrt{N}\sqrt{\sum \|H_{ij}\|^2}}$$

$$H|_K = C\sqrt{N}\sqrt{\sum_{i,j} \|H_{ij}\|^2_{L^2(K)}}$$ (on each element)

then can apply, e.g. advancing front technique with this spacing function etc.

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3. <u>A posteriori adaptivity</u>: here the strategy considered is based on estimation of the residual operator

$$R: \quad \langle R, v\rangle = F(v) - a(u_h, v)$$
$$= a(u - u_h, v)$$

Now, continuity/coercivity

$$\Rightarrow \alpha \|u - u_h\|_V \leq \|R\|_{V'} \leq M\|u - u_h\|_V$$

(i.e. operator norm $\sup_{v \neq 0} \frac{\langle R, v\rangle}{\|v\|}$),

Now take $\langle R, v\rangle = F(v - v_h) - a(u_h, v - v_h)$

(as $a(u - u_h, v) = a(u - u_h, v - v_h)$ by Galerkin orthogonality

$$= \underbrace{a(u, v - v_h)} - a(u_h, v - v_h)$$

$F(v-v_h)$ as u is solution.)

(consider case of Poisson problem)

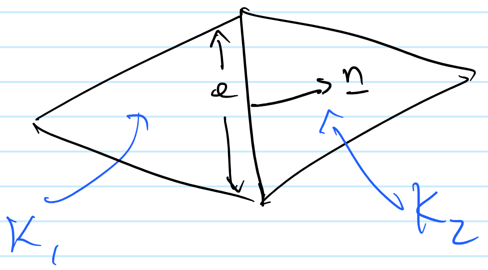
$$\overset{\neq 0}{\langle R, v\rangle} = \int_\Omega f(v-v_h) + \sum_K \int_K \nabla^2 u_h \,(v-v_h)$$

$$- \sum_K \int_{\partial K} \frac{\partial u_h}{\partial n}(v-v_h)$$

$$= \sum_K \int_\Omega (f+\nabla^2 u_h)(v-v_h) - \underbrace{\sum_K \int_{\partial K} \frac{\partial u_h}{\partial n}(v-v_h)}$$

To estimate ... define the jump of $\frac{\partial u_h}{\partial n}$ across an edge e

$$\left[\frac{\partial u_h}{\partial n}\right]_e = \left(\nabla u_h\big|_{K_1} - \nabla u_h\big|_{K_2}\right)\cdot \underline{n}$$



we can then define generalized jump by taking $\left[\frac{\partial u_h}{\partial n}\right]_e = 0$ if e is on boundary:

then

$$\sum_K \int_{\partial K} \frac{\partial u_h}{\partial n}(v-v_h) = \sum_{K_i} \frac{1}{2}\int_e \left[\frac{\partial u_h}{\partial n}\right]_e (v-v_h)$$

$$\sum_K \int_{\partial K} \frac{\partial u_h}{\partial n}(v - v_h) = \sum_{\substack{K, \\ e \in K}} \frac{1}{2}\int_e \left[\frac{\partial u_h}{\partial n}\right]_e (v - v_h)$$

$\frac{1}{2}$ as each internal edge appears twice.

$$\Rightarrow \langle R, v \rangle \leq \sum_K \Big(\| f + \nabla^2 u_h \|_{L^2(K)} \| v - v_h \|_{L^2(K)} + \frac{1}{2} \left\| \left[\frac{\partial u_h}{\partial n}\right] \right\|_{L^2(\partial K)} \| v - v_h \|_{L^2(\partial K)} \Big)$$

(Cauchy-Schwarz)

<u>Interpolation</u>: find a linear interpolant of v for estimation

→ Lagrange estimate has problems if v not continuous

→ can use, e.g. Clément interpolation, obeying

$$\| v - R_h v \|_{L^2(K)} \leq C h_K |v|_{H^1(\bar{K})}$$

$$\| v - R_h v \|_{L^2(\partial K)} \leq C h_K^{1/2} \| v \|_{H^1(\bar{K})}$$

(here $\bar{K}$ is union of all elements sharing an edge or vertex with K)

applying with Cauchy-Schwarz:

$$\langle R, v \rangle^2 \leq C \Big(\sum_K \rho_K(u_h)^2 \Big) \Big(\sum_K \| v \|^2_{H^1(\bar{K})} \Big).$$

$$\langle R, v\rangle \leq C\left(\sum_K \rho_K(u_h)^2\right)\left(\sum_K \|v\|^2_{H^1(\tilde K)}\right).$$

$$\rho_K(u_h) = h_K \|f + \nabla^2 u_h\|_{L^2(K)} + \frac{1}{2} h_K^{1/2} \left\| \left[\frac{\partial u_h}{\partial n}\right]\right\|_{L^2(\partial K)}$$

$\leq n \|v\|_{H^1(\Omega)}$

max no. of elements in $\tilde K$
(bounded by regularity assumption independent of h)

hence $\|R\|^2_{V'} \leq C n \sum_K \rho_K(u_h)^2$

and as $\alpha \|u - u_h\|_H \leq \|R\|^2_{V'}$, can use

[illegible] as estimate.