Linear elasticity

fredag 6. oktober 2017 08:53

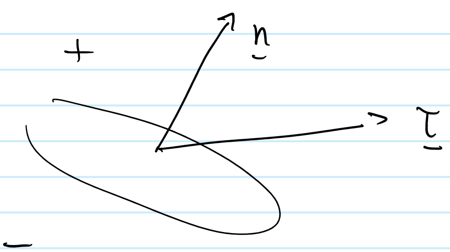
1. Consider a solid body of density ρ.

It is acted on by

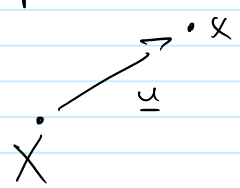
i) body forces $\underline{F}$: long range, proportional to volume (eg gravity)

ii) surface forces: short range, proportional to surface area.

Let $\underline{n}\,dS$ be a surface element at the point $(\underline{x}, t)$. Let $\underline{\tau}\,dS$ be the force exerted by the solid on one side of the element (+ve oriented) on the other (−ve)



2. Equation of motion: Let $u(x,t)$ be the displacement of a particle at time t



Momentum conservation:

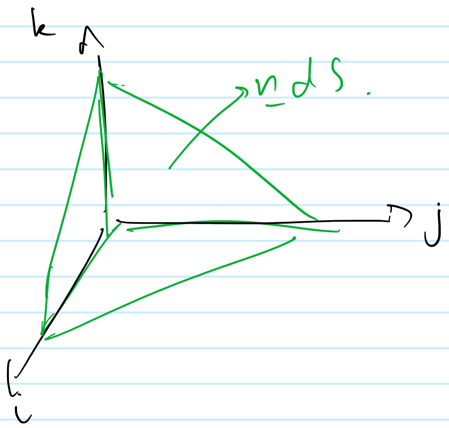
$$\underbrace{\frac{d}{dt}\int_V \rho \frac{\partial u}{\partial t}\, dV}_{O(\varepsilon^3)} = \underbrace{\int_V \rho F\, dV}_{= O(\varepsilon^3)} + \underbrace{\int_S \tau\, dS}_{O(\varepsilon^2)}$$

$O(\varepsilon^2)$ $\qquad = O(\varepsilon^3) \qquad O(\varepsilon^2)$

$\Rightarrow$ as $\varepsilon \to 0$, we must balance forces,

$$\text{i.e.} \quad \lim_{\varepsilon \to 0} \int_S \tau \, dS = 0$$

Consider the tetrahedron



Now $\quad \tau(n)\, dS + \tau(-i)\, i \cdot n \, dS + \tau(-j)\, j \cdot n \, dS$

$$+ \tau(-k)\, k \cdot n \, dS = 0$$

ie $\quad \tau(n) = \tau(i)\, i \cdot n + \tau(j)\, j \cdot n + \tau(k)\, k \cdot n$

$$\Rightarrow \quad \tau(n) = \sigma \cdot n$$

for a cartesian tensor of rank 2 σ,
called the stress tensor.

$\to$ Return to the momentum equation:

$$\frac{d}{dt} \int_V \rho \frac{\partial u}{\partial t} \, dV = \int_V \rho F \, dV + \int_S \sigma n \, dS.$$

small displacements

$$\Rightarrow \int_V \rho \frac{\partial^2 u}{\partial t^2}\, dV = \int_V \rho F\, dV + \underbrace{\int_V \nabla \cdot \sigma\, dV}_{\text{used divergence theorem.}}$$

holds on all V

$$\Rightarrow \boxed{\rho \frac{\partial^2 u}{\partial t^2} = \rho F + \nabla \cdot \sigma.}$$

Our next task is to relate $\sigma = \sigma(u)$, which we will accomplish by a discussion of strain. First:

a) $\sigma_{ij} = \sigma_{ji}$, symmetry of stress:

Examine angular momentum:

$$\frac{d}{dt}\int_V \rho \underline{x} \times \frac{\partial u}{\partial t}\, dV = \int_V \rho \underline{x} \times \underline{F}\, dV + \underbrace{\int_S \underline{x} \times \underline{T}\, dS.}_{\text{again, vanishes as } \varepsilon \to 0.}$$

$$= \int_S \varepsilon_{ijk}\, x_j\, \sigma_{km}\, n_m\, dS$$

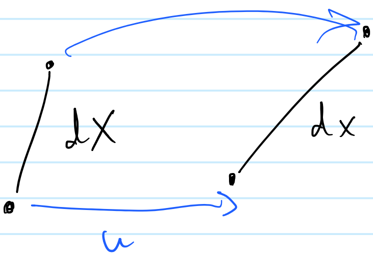
$$= \int_V \varepsilon_{ijk} \frac{\partial}{\partial x_m}(x_j \sigma_{km})\, dV$$ (divergence theorem)

$$= \underbrace{\int_V \varepsilon_{ijk} x_j \frac{\partial \sigma_{km}}{\partial x_m}\, dV}_{O(\varepsilon^4)} + \underbrace{\int_V \varepsilon_{ijk} \sigma_{kj}\, dV}_{O(\varepsilon^3)}$$

$\Rightarrow \quad \varepsilon_{ijk} \sigma_{kj} = 0, \quad$ ie $\quad \sigma_{ij} = \sigma_{ji}$

---

3. Strain: here we consider deformation of line elements (not due to rotation).



where X is initial position, $x(X, t)$ traces evolution of X.

NB: The distinction between working with big X (Lagrangian) and little x (Eulerian) is in general important, but can be ignored for small displacements.

In general, write eg. $\quad dx_i = \dfrac{\partial x_i}{\partial X_j} dX_j$

$$\Rightarrow \quad (dx)^2 - (dX)^2 = \left(\frac{\partial^2 x_k}{\partial x_i\, \partial x_j} - \delta_{ij} \right) dX_i\, dX_j .$$

and using $u = x - X$, have

$$= \frac{1}{2}\underbrace{\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i} + \frac{\partial^2 u_k}{\partial X_i \partial X_j} \right)}_{\text{finite strain tensor.}} dX_i \, dX_j$$

under the assumption $|\nabla u| \ll 1$,

here $\simeq \frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i} \right) = \frac{1}{2}(\nabla u + \nabla u^T)$

('infinitesimal strain tensor')

(start to omit x, X as $u(x,t) \simeq u(X,t)$ etc.)

i.e. $\varepsilon = \frac{1}{2}(\nabla u + \nabla u^T)$

---

4. Properties of solids: there exists a generalized Hooke's law relating stress and strain in linear elastic solids:

$$\sigma = C : \varepsilon$$

ie $\sigma_{ij} = C_{ijkl}\,\varepsilon_{kl}$, for some C_{ijkl}, a property of the material.

$\rightarrow$ in general, 81 parameters!!

→ assume that the material is isotropic

$\Rightarrow \quad C_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu \delta_{ik} \delta_{jl} + \mu' \delta_{il} \delta_{jk}$

(the most general rank 4 isotropic tensor!)

symmetry of $\sigma \Rightarrow \mu = \mu'$

hence $\quad C = \lambda \delta_{ij} \delta_{kl} + \mu (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk})$

$$\Rightarrow \quad \boxed{\sigma_{ij} = \lambda \delta_{ij} e_{kk} + 2\mu e_{ij}}$$

here λ, μ are Lamé moduli of the material

(there are various measurements for different physical situations, eg shear modulus, bulk modulus, Young's modulus, Poisson's ratio etc.)

5\. Collecting the above, we obtain the equations of linear elasticity:

$$\left.\begin{aligned} \rho \frac{\partial^2 u}{\partial t^2} &= \nabla \cdot \sigma + \rho F \\ \varepsilon &= \tfrac{1}{2}(\nabla u + \nabla u^T) \end{aligned}\right\}$$

$$\varepsilon = \tfrac{1}{2}(\nabla u + \nabla u^{T})$$

$$\sigma = C : \varepsilon$$

(typically $C = C(\lambda, \mu)$).

with appropriate boundary conditions.

$\downarrow$ towards a weak formulation.

(Note that the equation looks like

$$\rho \frac{\partial^2 u}{\partial t^2} = (\lambda + \mu)\nabla(\nabla \cdot \underline{u}) + \mu \nabla^2 \underline{u} + \rho F$$

e.g. in the static case, for instance consider buildings,

$$(\lambda + \mu)\nabla(\nabla \cdot \underline{u}) + \mu(\nabla^2 \underline{u}) + \rho F = 0$$

this begins to look more familiar).

6. Weak form (exercise):

$$\int_\Omega \bar{\varepsilon}(v)^T C\, \bar{\varepsilon}(u) = \int_\Omega v^T f + \int_{\partial\Omega} v^T \sigma\, n\, ds$$

here $\bar{\varepsilon}$ is the vectorized stress, eg has components

$$\begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{12} \\ \varepsilon_{13} \\ \varepsilon_{21} \end{pmatrix}$$

and C has been similarly

and C has been similarly transformed into a matrix.

$$\begin{pmatrix} \vdots \\ \varepsilon_{21} \\ \vdots \end{pmatrix}$$

NB: an important difference is that test functions are multidimensional: typically take $\varphi^1(x) = \begin{pmatrix} \varphi(x) \\ 0 \\ 0 \end{pmatrix}$ or

$$\varphi^2(x) = \begin{pmatrix} 0 \\ \varphi(x) \\ 0 \end{pmatrix} \text{ etc.}$$

$\Rightarrow$ d test functions for each node etc.