Condition number

torsdag 5. oktober 2017 12:26

In previous lectures, we have derived error bounds of a type

$$\|u-u_h\|_{H^1} \leq C h^r |u|_{H^{r+1}}$$

This suggests that we can make the error arbitrarily small by taking h large enough.
(assuming we can solve the resulting systems)

In practice, this is not usually the case, due to the ill-conditioning of the stiffness matrix.

1. Condition number. Let $\|\cdot\|$ be the l^2 operator norm

$$\|A\| = \sup_{\|v\| \leq 1} \|Av\|, \quad \text{where } \|\cdot\| \text{ is Euclidean vector norm.}$$

We then define the condition number of a matrix A to be

$$K_2(A) = \|A\| \, \|A^{-1}\|$$

(if A is singular, $K(A) := \infty$).

The condition number is a measure of how relative errors are amplified under the solving of a linear system

Indeed, suppose u solves the system

$Au = b$, and u' solves $Au' = b'$.

$$\Rightarrow \frac{\|u - u'\|}{\|u\|} \leq K(A) \frac{\|b - b'\|}{\|b\|}.$$

[Proof: let $e = b - b'$. then

$$\left(\frac{\|A^{-1}e\|}{\|A^{-1}b\|} \Big/ \frac{\|e\|}{\|b\|} \right)$$

$$= \frac{\|A^{-1}e\|}{\|e\|} \frac{\|b\|}{\|A^{-1}b\|}$$

$$= \frac{\|A^{-1}e\|}{\|e\|} \frac{\|Ab\|}{\|b\|}.$$

$$\leq \|A^{-1}\| \cdot \|A\| = K(A).$$].

2. Note that $\|A\|^2 = \max \sigma(A)$, the largest singular value of A (eigenvalue of $A^T A$)

$\Rightarrow$ for symmetric A, $\|A\| = \rho(A)$, the spectral radius of A.

$\Rightarrow$ for symmetric A, (positive definite) $K_2(A) = \dfrac{\lambda_{max}(A)}{\lambda_{min}(A)}$,

3. Condition number of stiffness matrix.

We will show that

$\kappa_2(A) = C(r)\, h^{-2}$, hence the matrix

becomes increasingly ill-conditioned as we refine the grid.

Indeed, suppose $a(\cdot,\cdot)$ is symmetric, coercive, so that the stiffness matrix A_h is symmetric, positive definite.

$\Rightarrow$ has +ve eigenvalues $\lambda_h \dots$

Let v_h be the function associated to eigenvector $\underline{v}_h$ of λ_h, ie $v_h(x) = \sum (\underline{v}_h)_i \varphi_i(x)$

then $Av = \lambda_h v$

$$\Rightarrow v^T A v = \lambda_h v^T v = \lambda_h \|v\|^2$$

ie $$\lambda_h = \frac{a(v_h, v_h)}{\|\underline{v}\|^2}.$$

hence

$$\alpha \|v_h\|^2_{H^1} \leq \lambda_h \|\underline{v}\|^2 \leq M \|v_h\|^2_{H^1}$$

It may be shown that see later!

It may be shown that

$$\boxed{c_1 h^d \|\underline{v}\|^2 \leq \|v_h\|^2_{H^1} \leq c_2 h^{d-2} \|\underline{v}\|^2}$$

→ Assuming the triangulation is regular enough.

$$\Rightarrow \quad \frac{\lambda_{max}}{\lambda_{min}} \leq c \cdot \frac{M}{\alpha} h^{-2}.$$

4. Consequences: suppose we are converging as h^n, but the condition grows as h^{-2}, we then have

$$\text{err} \sim h^n + h^{-2}\varepsilon, \quad \text{where } \varepsilon \text{ is the source of error in load.}$$

(relative)

$\Rightarrow$ suppose Gaussian quadrature sufficiently accurate. (eg. let ε be floating point error)

We attempt to find the minimum error.

$$\Rightarrow \text{err}' = n h^{n-1} - 2h^{-3}\varepsilon$$

$$= h^{n-1}\left(1 - \frac{2}{n} h^{-2-n} \varepsilon\right)$$

$$\text{ie} \quad h \sim \varepsilon^{\frac{1}{2+n}}$$

$$\text{when err} \sim \varepsilon^{\frac{n}{2+n}} + \varepsilon^{1-\frac{2}{2+n}} \quad \text{etc.}$$

when err $\sim \varepsilon$

$$\sim \varepsilon^{\frac{n}{2+n}}$$

e.g.

$n=$	1	2	3	4
$\varepsilon^{\wedge}$	$1/3$	$1/2$	$3/5$	$2/3$

etc.

best order of magnitude of error wrt machine accuracy.

5. Note that ill-conditioning also has negative effects on the speed of convergence of iterative schemes. (we will discuss this later).

a preconditioner P is frequently employed,

ie P : $\kappa_2(P^{-1}A) < \kappa_2(A)$.

typically incorporated in iterative scheme. Many different strategies in practice.

---

6. Postscript: the following two Lemmas were used in the derivation of the condition number, via the green box.

condition number, via the green box.

1. $\|\nabla v_h\|_{L^2} \leq C h^{-1} \|v_h\|_{L^2}$ (again, need regularity of the triangulation here).

$\left(\Rightarrow \|v_h\|_{H^1} \leq C h^{-1} \|v_h\|_{L^2} \right)$

2. $C_1 h^d |\underline{v}|^2 \leq \|v_h\|_{L^2}^2 \leq C_2 h^d |\underline{v}|^2.$

(an inequality like this may have been anticipated by many!)

As noted, the latter inequality allow us to estimate L_2 errors using the readily computable vector norm.

The assumption is quasi-uniformity, i.e.

$$\exists C: \quad \min_K h_K \geq C \cdot \underbrace{\max_K h_K}_{h}.$$

Proof of 2: quasi-u $\Rightarrow$ number of edges intersecting a given node is uniformly bounded.

Let $\hat{v} = \sum_{i \in K} v_i \hat{\varphi}_i$, a function on a reference element.

Then let $\int_{\hat{K}} \hat{v}^2$ ... as a continuous then let $\psi(v) = \dfrac{\int_{\hat{K}} \hat{v}^2}{|v|^2}$; as a continuous function on compact $\{v : |v|^2 = 1\}$, it attains its bounds, c_1 and c_2, but is invariant under scaling

$$\Rightarrow \quad c_1 |v|^2 \leq \int_{\hat{K}} \hat{v}^2 \leq c_2 |v|^2$$

but $\displaystyle\int_K v^2 = \int_{\hat{K}} \hat{v}^2 \frac{|K|}{|\hat{K}|}$.

regularity $\Rightarrow \quad c_3 h^d \leq \dfrac{|K|}{|\hat{K}|} \leq c_4 h^d$

from which the result follows.