Error analysis 3

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1. Assume we have discretized a parabolic PDE by a Galerkin method to find an ODE

$$M u' + A u = f$$

We consider the family of methods indexed by a parameter $\theta \in [0,1]$

$$M \frac{u^{k+1} - u^k}{\underbrace{\Delta t}_{= t_{k+1} - t_k}} + A\left(\theta u^{k+1} + (1-\theta) u^{k-1}\right) = \theta f^{k+1} + (1-\theta) f^k$$

$\theta = 0$: (Forward) Euler

$\theta = 1$: backward / implicit Euler

$\theta = 1/2$: trapezoidal rule (Crank-Nicolson)

In practice, $\underbrace{\left(\frac{M}{\Delta t} + \theta A\right)}_{\text{symmetric, positive definite if } a(\cdot,\cdot) \text{ is symmetric.}} u^{k+1} = g(u^k)$

Assume a constant timestep, the matrix in front of u^{k+1} is constant

-> in symmetric case, can use Cholesky factoring $= LL^T$

(then at each step solve $Ly = g(u^k)$)

$L^T u^{k+1} = y$)

→ more generally, use LU factorization (with pivots)

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2. We now examine stability of the temporal discretization. Assume that $a(\cdot,\cdot)$ is symmetric

⇒ we can form an eigenbasis for V from $w \in V$: $\quad a(w,v) = \lambda (w,v) \quad \forall v$

⇒ in Galerkin formulation, eigenvectors obey $\quad Aw = \lambda_H M w$

(i.e. eigenvectors of $M^{-1}A$)

⇒ Here we also have an orthogonal eigenbasis (all such generalized eigenvalue problems with A symmetric, M +ve definite have this property)

We will now write $u_h^k = \sum_j u_j^k w_h^j(x)$

Now consider the homogeneous equation with $f(x,t) = 0$

Multiply (discretized) ODE by w_i^T:

$$w_i^T w_i M \frac{u_j^{k+1} - u_j^k}{} + w_i^T A w_j \left(\theta u_j^{k+1} + (1-\theta) u_j^k \right) = 0$$

$$\underbrace{w_i^T w_j}_{=\delta_{ij}} M \frac{u_j^{k+1} - u_j^k}{\Delta t} + \underbrace{w_i^T A w_j}_{=\delta_{ij}\lambda_h^i} \left(\theta u_j^{k+1} + (1-\theta) u_j^k \right) = 0$$

i.e. $$\frac{u_i^{k+1} - u_i^k}{\Delta t} + \left[\theta u_i^{k+1} + (1-\theta) u_i^k \right] \lambda_h^i = 0$$

Solve for u_i^{k+1}

$$\Rightarrow \quad u_i^{k+1} = u_i^k \Bigg(\underbrace{\frac{1 - (1-\theta)\lambda_h^i \Delta t}{1 + \theta \lambda_h^i \Delta t}}_{\text{Need } < 1 \text{ for stability}} \Bigg)$$

in particular, it can be verified that we require:

$$\Delta t < \frac{2}{(1-2\theta)\lambda_h^i} \quad \text{if } \theta < \frac{1}{2}$$

whilst if $\theta \geq \frac{1}{2}$ the inequality is always satisfied

$\rightarrow$ Moreover, it may be shown that

$\max_i \lambda_h^i \simeq c h^2$, hence with $\theta < \frac{1}{2}$

we will require $\Delta t \leq C(\theta) h^2$

(as per finite difference schemes) This is very restrictive)

3. Convergence: for sufficiently regular exact solution and coefficients, we have

(Theorem):

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$$\| u(t^n) - u_h^n \|_{L^2}^2 + 2\alpha \Delta t \sum_{k=1}^{n} \| u(t^k) - u_h^k \|_{H^1}^2$$

$$\leq C(u_0, f, u) \left(\Delta t^2 + h^{2r} \right)$$

in case $\theta = \frac{1}{2}$ (trapezoidal rule) improves to Δt^4.

[Proof omitted].

<u>Comments</u>: Although it is unconditionally stable and of higher order than other θ methods, you may find that the trapezoidal rule still performs worse than backward Euler in many cases. It can sometimes allow, eg bounded but rapidly oscillating solutions that settle down slowly. Experiment!

(the worst behaviours occur with large imaginary eigenvalues, which we fortunately avoid).