# 1d galerkin

onsdag 23. august 2017 11.02

We begin with the equation

$$\begin{cases} -u''(x) = f(x), & 0 < x < 1, \\ u(0) = u(1) = 0 \end{cases}$$

Then multiply by test function $v$:

$$-u''v = fv$$

Integrate:

$$\underbrace{-\int_0^1 u''v \, dx}_{= \int_0^1 u'v' \, dx - \cancel{[u'v]_0^1}} = \int_0^1 fv \, dx$$

$$= \int_0^1 u'v' \, dx - \cancel{[u'v]_0^1} \to 0$$

consider $v$:
$v(0) = v(1) = 0$

$$\Rightarrow \boxed{\int_0^1 u'v' \, dx = \int_0^1 fv \, dx \quad \forall v \in V.}$$

weak formulation.

here look for solutions in same space.

A little care is required: what is the correct space for $V$ etc.?

↳ define $H^1(0,1)$, space such that functions and weak derivative square integrable

and $H^1_0(0,1)$, where $v(0) = v(1) = 0$

want square integrable because $u', v'$ sq-int. $\Rightarrow u'v'$ int.

$\Rightarrow$ $u'v'$ int.

Weak derivative:

$$\int_0^1 u'v\,dx = \cancel{[uv]_0^1}^{\,0} - \int_0^1 uv'\,dx \quad \forall v$$

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Galerkin scheme: solve $u \in V_h$ :

$$\int_0^1 u'v' = \int_0^1 fv \qquad \forall v \in V_h.$$

where $V_h$ is some finite-dimensional function space.

claim: reduces to linear system:

let $u = u_i \varphi_i$, $\qquad v = v_j \varphi_j$

$\Rightarrow$ $u' = u_i \varphi_i'$ $\qquad v' = v_j \varphi_j'$

then
$$\int_0^1 u'v' = u_i \left( \int_0^1 \varphi_i' \varphi_j' \right) v_j$$
$$= v^T A u$$

$$\int_0^1 fv = v_i \int f \varphi_i = v^T F$$

i.e. $\quad v^T A u = v^T F \qquad \forall v$

$$\Rightarrow \quad Au = F$$

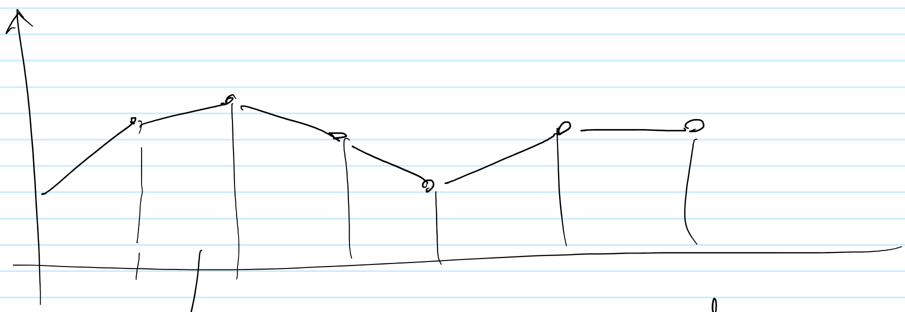
$$[A_{ij}] = \int_0^1 \varphi_i' \varphi_j' \, dx \qquad\qquad F_i = \int_0^1 F \varphi_i \, dx$$

"Stiffness matrix" $\qquad\qquad$ "Load vector"

Finite elements: pick $\varphi_i$ localized, so that $A$ is sparse

e.g. let $V_h$ be piecewise polynomials of a given degree,

degree 1:

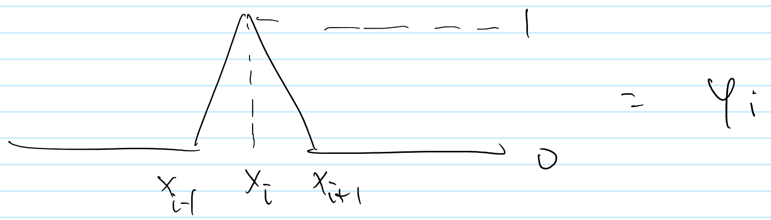


etc.

each interval is called an element.

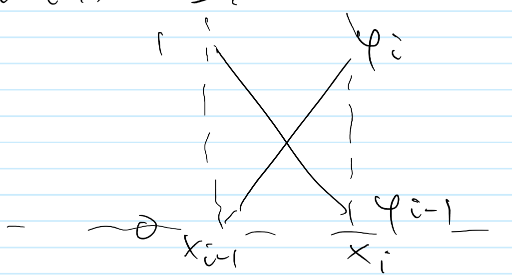
1d case: Lagrange basis: $\varphi_i(x_j) = \delta_{ij}$.

i.e.




(remember polynomial interpolation... )

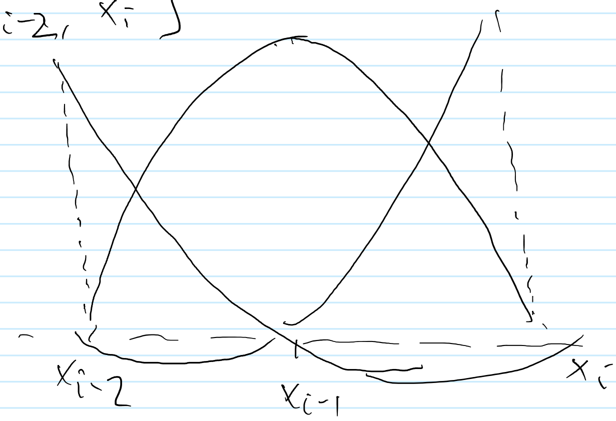
usually, it is better to think of basis fns only when restricted to an element, ie on $[x_{i-1}, x_i]$, we have



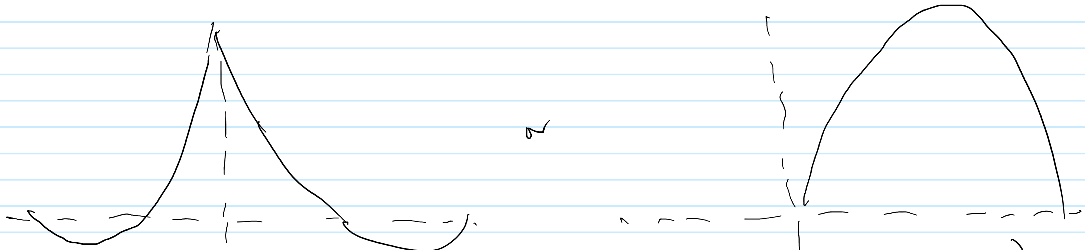

(return to this idea in construction of A)

2d case: have interior point (need 3 values to determine quadratic)

on $[x_{i-2}, x_i]$




i.e. basis functions look like either




(even $i$, $\varphi_i$) (odd $i$, $\varphi_i$)

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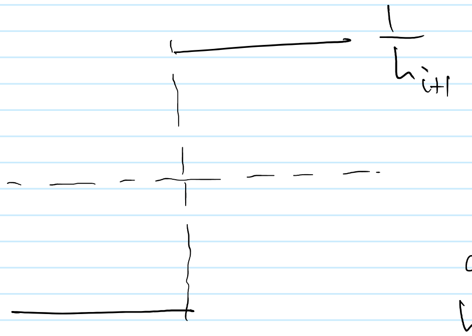
Constructing stiffness. Consider first linear case.

Now

$\varphi_i'$ :




$h_i = x_i - x_{i-1}$

assume for now $h_i = h$ constant.

idea: $\displaystyle\int_0^1 \varphi_i' \varphi_j' = \sum_{\text{elements}} \int_0^1 \varphi_i'\big|_{el} \varphi_j'\big|_{el} = \sum_{i=1}^{N} \int_{x_{i-1}}^{x_i} \varphi_i' \varphi_j'$

on element $T_i$, only $\varphi_i$, $\varphi_{i+1}$ are non zero,

$\varphi_i' = -\frac{1}{h}, \quad \varphi_{i+1}' = \frac{1}{h}.$

ie. $\varphi_i' \varphi_{i+1}' = -\frac{1}{h^2}, \quad \varphi_i' \varphi_i' = \varphi_{i+1}' \varphi_{i+1}' = \frac{1}{\ldots}$

i.e. $\varphi_i' \varphi_{i+1}' = -\frac{1}{h^2}$, $\varphi_i' \varphi_i' = \varphi_{i+1}' \varphi_{i+1}' = \frac{1}{h^2}$

as $\int_{x_{i-1}}^{x_i} dx = h$, have $T_i$ contributes

$$\frac{1}{h}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \longleftarrow \text{insert at } A_{ij}$$

(all other terms give zero contribution in this element)

e.g. 3 elements,

$$A_{ij} = \frac{1}{h}\begin{pmatrix} 1 & -1 & & \\ -1 & 1 & & \\ & & 0 & \\ & & & 0 \end{pmatrix} + \begin{pmatrix} 0 & & & \\ & 1 & -1 & \\ & -1 & 1 & \\ & & & 0 \end{pmatrix} + \ldots$$

$$= \frac{1}{h}\begin{pmatrix} 1 & -1 & & \\ -1 & 2 & -1 & \\ & -1 & 2 & -1 \\ & & -1 & 1 \end{pmatrix}$$

(Exercise: 2d case, instead we get a 3d sub-block.

$$\frac{1}{3h}\begin{pmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7 \end{pmatrix} \longrightarrow \text{insert with top-left at each odd diagonal entry})$$

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Constructing load vector: work elementwise again

$$\int_0^1 f(x)\varphi_i(x)\,dx = \sum_i \int_{x_{i-1}}^{x_i} f(x)\,\varphi_i(x)\,dx.$$

or $x_{i-2}$ to $x_i$ in quadratic case, etc.

i.e. at each linear element, add vector

$$\left( \int_{x_{i-1}}^{x_i} f(x) \frac{x - x_i}{h}\,dx \right)$$

case, etc.

$$\begin{pmatrix} \displaystyle\int_{x_{i-1}}^{x_i} f(x)\,\frac{x - x_i}{-h}\,dx \\ \displaystyle\int_{x_{i-1}}^{x} f(x)\,\frac{x - x_{i-1}}{h}\,dx \end{pmatrix}$$

usually must compute by quadrature

eg $$\int_a^b f(x)\,dx \simeq \frac{b-a}{2} \sum_i w_i\, f\left(\frac{b-a}{2} x_i + \frac{a+b}{2}\right)$$

at carefully selected quadrature nodes $x_i$ and weights $w_i$

in quadratic case, will add three-vectors of

eg $\displaystyle\int_{x_{i-2}}^{x_i} f(x)\, L_1(x)\, dx$, $\quad L_1(x) = \dfrac{(x - x_i)(x - x_{i-1})}{(x_{i-2} - x_i)(x_{i-2} - x_{i-1})}$

etc.

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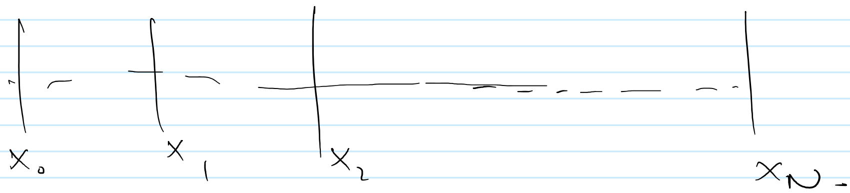
Boundary conditions: so far, our space of functions $V_h$




we must enforce $v(0) = v(1) = 0$, to satisfy bcs.

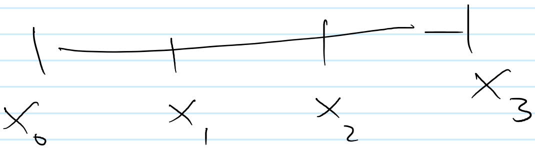
$\Rightarrow \quad u_0 = u_N = 0$

all rows/columns of $A_{ij}$, $F_i$ corresponding must be deleted.

eg if $(h = 1/3)$

$$A = 3\begin{pmatrix} 1 & -1 & & \\ -1 & 2 & -1 & \\ & -1 & 2 & -1 \\ & & 1 & 1 \end{pmatrix}$$

in 3 element case




$$A \longrightarrow 3\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

eg $F = \begin{pmatrix} 1/4 \\ 3/7 \\ -1/6 \\ 1/8 \end{pmatrix} \longrightarrow F = \begin{pmatrix} 3/7 \\ -1/6 \end{pmatrix}$

and will solve system

$$3\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3/7 \\ -1/6 \end{pmatrix}.$$

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