Exercise set 2, TMA4220

September 15, 2015

1. **The Poincaré inequality.** In this exercise you will prove Poincaré's inequality: If $\Omega \subset \mathbb{R}^d$ is an open, bounded domain then there exists a constant $C > 0$ such that

$$\|u\|_{L^2(\Omega)} \leqslant C|u|_{H^1(\Omega)} \tag{1}$$

for every $u \in H^1(\Omega)$. Here, $|u|_{H^1(\Omega)}$ denotes the H^1 seminorm

$$|u|_{H^1(\Omega)} := \|\nabla u\|_{L^2(\Omega)} = \left(\int_\Omega \sum_{i=1}^d \left|\frac{\partial u}{\partial x_i}\right|^2 \, dx\right)^{1/2}.$$

Informally, *in H_0^1 on bounded domains, you can bound the size of a function by its derivative.*

(a) We assume first that $\Omega = [-M, M]^2$, the box in $\mathbb{R}^2$ with center 0 and side lengths $2M$ (for some $M > 0$). Show that there is a $C > 0$ such that for all $u \in H_0^1(\Omega)$,

$$|u(x)|^2 \leqslant C \int_{-M}^{M} \left|\frac{\partial u}{\partial x_1}(y_1, x_2)\right|^2 \, dy_1 \qquad \forall\, x = (x_1, x_2) \in \Omega.$$

Hint: Use the fundamental theorem of calculus and Cauchy's inequality.

(b) Integrate over x_2 and show that

$$\int_{-M}^{M} |u(x_1, x_2)|^2 \, dx_2 \leqslant C \int_\Omega \left|\frac{\partial u}{\partial x_1}(y)\right|^2 \, dy \qquad \forall\, x_1 \in [-M, M].$$

(c) Now integrate over x_1 and conclude with (1).

(d) If $\Omega \subset \mathbb{R}^2$ is a general open, bounded domain, you can find some $M > 0$ such that $\Omega \subset [-M, M]^2$ (why? Can you come up with an explicit candidate for such an M?). For a function $u \in H_0^1(\Omega)$, find some function $\bar{u} \in H_0^1([-M, M]^2)$ such that $u = \bar{u}$ in Ω and

$$\|\bar{u}\|_{L^2([-M,M]^2)} = \|u\|_{L^2(\Omega)} \qquad \text{and} \qquad |\bar{u}|_{H^1([-M,M]^2)} = |u|_{H^1(\Omega)}.$$

Explain why this implies that the Poincaré inequality also holds on Ω.

Remarks:

- The Poincaré inequality on bounded domains $\Omega \subset \mathbb{R}^d$ is shown in an analogous fashion.
- Note that the constant C grows (linearly) as the box becomes larger.
- If $u \in H_0^k(\Omega)$ for any $k \geqslant 1$ then $\frac{\partial u}{\partial x_i} \in H_0^{k-1}(\Omega)$ for any $i = 1, \ldots, d$, and more generally, $D^\alpha u \in H_0^{k-|\alpha|}$ for any multiindex α of size $|\alpha| \leqslant k$. Thus, we can iterate Poincaré's inequality and find that there is a constant $C > 0$ such that

$$\|u\|_{L^2(\Omega)} \leqslant C|u|_{H^1(\Omega)} \leqslant \cdots \leqslant C|u|_{H^k(\Omega)} \qquad \forall\, u \in H_0^k(\Omega),$$

where $|u|_{H^k(\Omega)}$ is the H^k-seminorm,

$$|u|_{H^k(\Omega)} := \left(\int_\Omega \sum_{\alpha:|\alpha|=k} |D^\alpha u|^2 \, dx\right)^{1/2}$$

(where the sum goes over all multiindices α of size equal to k).

2. **Exercises on elliptic problems.**
For each of the following problems, find a weak formulation of the PDE and choose an appropriate test/trial space V where u and v live. Show that the conditions of the Lax-Milgram theorem are satisfied, thus proving that there exists a unique (weak) solution of the PDE. Unless otherwise stated, $\Omega \subset \mathbb{R}^d$ is an open, bounded, connected domain and $f \in L^2(\Omega)$.

   (a) The biharmonic equation:
$$\begin{cases} \Delta^2 u = f & \text{in } \Omega, \\ u = \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega \end{cases} \tag{2}$$
   where $\Delta^2 u = \Delta(\Delta u)$.

   (b) The convection-diffusion equation:
$$\begin{cases} -\Delta u + a \cdot \nabla u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{3}$$
   where $a = a(x) : \Omega \to \mathbb{R}^d$ is a given differentiable function, the *velocity field*, satisfying $\nabla \cdot a(x) = 0$ for all x.

   (c) The nonhomogeneous Laplace equation:
$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega \end{cases} \tag{4}$$
   where $g : \partial\Omega \to \mathbb{R}$ is a given function.
   Hint: We may assume that Ω and g are "nice enough" so that there exists a function $\bar{g} : \overline{\Omega} \to \mathbb{R}$ such that $\bar{g}\big|_{\partial\Omega} = g$.

   (d) Laplace's equation with mixed, nonhomogeneous boundary conditions:
$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = g & \text{on } \Gamma_D, \\ \frac{\partial u}{\partial n} = h & \text{on } \Gamma_N \end{cases} \tag{5}$$
   where Γ_D, Γ_N are disjoint parts of the boundary of Ω such that $\overline{\Gamma_D} \cup \overline{\Gamma_N} = \partial\Omega$, and $g : \Gamma_D \to \mathbb{R}$ and $h : \Gamma_N \to \mathbb{R}$ are given functions.

3. **The Neumann problem**
Consider the Laplace equation with Neumann boundary conditions,
$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases} \tag{6}$$

   (a) Find a condition on f that must be satisfied for (6) to make sense.
   Hint: Integrate the PDE over $x \in \Omega$ and use the divergence theorem.

   (b) There is no unique solution of (6). Why?

   (c) Consider the one-dimensional problem
$$\begin{cases} -\frac{d^2}{dx^2} u = f & \text{in } (a, b), \\ \frac{du}{dx} = 0 & \text{at } x = a, b. \end{cases} \tag{7}$$
   Find an explicit solution of (7) by using the fundamental theorem of calculus. Show that we can make the solution unique by additionally fixing the value of u at one of the endpoints $x = a$ or $x = b$.