Exercise set 1, TMA4220

August 21, 2015

Notation:

- For a set Ω, we write $L^2(\Omega) = \{v : \Omega \to \mathbb{R}$ such that $\|v\|_{L^2(\Omega)} < \infty\}$, where $\|v\|_{L^2(\Omega)} = \left(\int_\Omega |v(x)|^2 dx\right)^{1/2}$. If $u, v \in L^2(\Omega)$ we write $(u,v) = \int_\Omega u(x)v(x)\,dx$. The Cauchy-Schwartz inequality says that $|(u,v)| \leqslant \|u\|_{L^2}\|v\|_{L^2}$.
- $C(\Omega)$ is the set of continuous functions defined in Ω. For $k \in \mathbb{N}$, $C^k(\Omega)$ is the set of continuous functions defined in Ω such that every derivative up to the k-th order is also continuous in Ω.
- If Ω is a bounded, open interval on $\mathbb{R}$ then $H^1(\Omega)$ is the set of functions $u \in C(\bar{\Omega})$ such that $\frac{du}{dx}$ is piecewise continuous and bounded. It may be shown that this definition is equivalent to

$$H^1(\Omega) := \{u \in L^2(\Omega) \text{ such that its weak derivative } \tfrac{du}{dx} \text{ lies in } L^2(\Omega)\}$$

  (see exercise 1.d for the definition of weak derivative).
- $H_0^1(\Omega)$ is the set of functions in $H^1(\Omega)$ that vanish (i.e. are zero) on the boundary of Ω.

Exercises:

1. **Exercises on Lebesgue and Sobolev spaces.**

   (a) For which $\alpha \in \mathbb{R}$ does the function $f(x) := |x|^\alpha$ lie in $L^2([-1,1])$? What about $L^2([1,\infty))$? What about $L^2(B_1(0))$, where $B_1(0) = \{x \in \mathbb{R}^2 : |x| < 1\}$ is the unit ball in $\mathbb{R}^2$?

   (b) If $D \subset \mathbb{R}$ is a closed, bounded subset of $\mathbb{R}$ and $f \in C(D)$, show that $f \in L^2(D)$.

   (c) Let $\Omega \subset \mathbb{R}^d$ be an open, bounded set and let $u \in C(\Omega)$, the space of continuous functions on Ω. Show that if $\int_\Omega uv\,dx = 0$ for all $v \in C(\Omega)$ then $u \equiv 0$ on Ω.

   (d) Let $\Omega \subset \mathbb{R}$ be some open interval. A *weak derivative* of a function $u : \Omega \to \mathbb{R}$ is a function $v : \Omega \to \mathbb{R}$ such that

   $$\int_\Omega u(x)\varphi'(x)\,dx = -\int_\Omega v(x)\varphi(x)\,dx$$

   for every $\varphi \in C_c^\infty(\Omega)$, the set of infinitely differentiable functions with compact support in Ω. Show that the weak derivative (if it exists) is unique. Show that if u is continuously differentiable (i.e. $u \in C^1(\Omega)$), then $\frac{du}{dx}$ is its weak derivative.

   (e) Let

   $$f_1(x) := \begin{cases} x & \text{if } 0 < x < 1 \\ 1 & \text{if } 1 \leqslant x < 2, \end{cases} \qquad f_2(x) := \begin{cases} x & \text{if } 0 < x < 1 \\ 2 & \text{if } 1 \leqslant x < 2 \end{cases}$$

   for $x \in \Omega := (0,2)$. Show that $f_1, f_2 \in L^2(\Omega)$. Show that $f_1 \in H^1(\Omega)$ by finding its weak derivative, and that $f_1 \notin H^2(\Omega)$. Show that $f_2 \notin H^1(\Omega)$.

2. **Classical and weak solutions.**
   Consider the equation

$$\begin{cases} -u'' + u = f & \text{in } \Omega := (0,1) \\ \frac{du}{dx}(0) = \frac{du}{dx}(1) = 0 \end{cases} \tag{1}$$

for some $f \in L^2(\Omega)$. We consider the following weak formulation of (1):

$$\text{find } u \in V \text{ such that } (u', v') + (u, v) = (f, v) \quad \forall\, v \in V, \tag{2}$$

where $V = H^1(\Omega)$. Show that if u is a weak solution of (1) (i.e., it satisfies (2)) and in addition $u \in C^2(\Omega)$, then u is a classical solution of (1) (i.e., it satisfies (1) pointwise). Note carefully that the Neumann boundary condition does *not* appear explicitly in the weak formulation (2) – it is a *natural* boundary condition.

3. **Finite difference and finite element methods for Poisson's equation.**
Consider the one-dimensional Poisson equation

$$\begin{cases} -u'' = f & \text{in } (0,1) \\ u(0) = u(1) = 0 \end{cases} \tag{3}$$

for some $f \in L^2((0,1))$. Write down a finite difference method for (3) on a uniform mesh, and compare with a $\mathbb{P}_1$ finite element method. What are the differences and similarities?

4. **Optimal rate of convergence for Poisson's equation.** *(Taken from C. Johnson, exercise 1.19.)*
Consider the model problem (3). We discretize the domain $\Omega = (0,1)$ into nodes $0 < x_1 < x_2 \cdots < x_N < 1$ with size $h = \max_{i=1,\dots,N+1}(x_i - x_{i-1})$ (here, we set $x_0 = 0$, $x_{N+1} = 1$). For each $i = 1, \dots, N$, let $G_i \in H_0^1(\Omega)$ be the solution of the weak formulation

$$\int_\Omega G_i' v' \, dx = v(x_i) \qquad \forall\, v \in H_0^1(\Omega) \tag{4}$$

(note that the right-hand side point x_i is fixed). It may be shown that G_i is given by

$$G_i(x) = \begin{cases} (1 - x_i)x & \text{for } 0 \leqslant x \leqslant x_i \\ x_i(1 - x) & \text{for } x_i < x \leqslant 1. \end{cases}$$

(G_i is called the *Green's function* for (3) and satisfies (formally) $-G_i'' = \delta_{x_i}$, where δ_{x_i} is the Dirac delta function at x_i.)

   (a) Consider a finite element approximation of (3) with the test space $V_h = X_h^1$. Show that in fact $G_i \in X_h^1$.

   (b) Let $u \in H_0^1$ be the weak solution of (3), and let $u_h \in X_h^1$ be the finite element approximation of (3). Then also $e := u - u_h$ lies in $H_0^1(\Omega)$, so we can let $v = e$ in (4). Show that in fact

   $$e(x_i) = (e', G_i') = 0 \qquad \forall\, i = 1, \dots, N.$$

   This remarkable fact – a special property of (3) – means that the finite element approximation is actually *exact* at the nodes $x_1, x_2, \dots, x_N$.

   (c) Show that, as a consequence of the accuracy of linear interpolation, we have

   $$\|u - u_h\|_{L^2(\Omega)} \leqslant Ch^2 \|u''\|_{L^2(\Omega)}$$

   for some $C > 0$. Note that this $O(h^2)$ accuracy is better than the $O(h)$ result coming from Cea's lemma.

5. **Programming exercise.**
In this exercise you will program a finite element code for the model problem (3). Although the assembly of the stiffness matrix A in this particular case can be done directly, you should structure your code using local stiffness matrices $A_{\alpha,\beta}^{(k)}$, the reference element $\hat{K}$, etc., as explained in class.

   (a) Write a Matlab function

   ```
   function I = quad1d(n, f)
   ```

   that computes the numerical approximation

   $$\int_0^1 f(x)\, dx \approx \sum_{q=1}^{n} w_q f(x_q)$$

n	x_q	w_q
1	$\frac{1}{2}$	1
2	$\pm\sqrt{\frac{1}{12}}+\frac{1}{2}$	$\frac{1}{2}$
3	$\frac{1}{2}$	$\frac{8}{18}$
	$\pm\sqrt{\frac{3}{20}}+\frac{1}{2}$	$\frac{5}{18}$
4	$\pm\sqrt{\frac{3}{28}-\frac{1}{7}\sqrt{\frac{6}{5}}}+\frac{1}{2}$	$\frac{18+\sqrt{30}}{72}$
	$\pm\sqrt{\frac{3}{28}+\frac{1}{7}\sqrt{\frac{6}{5}}}+\frac{1}{2}$	$\frac{18-\sqrt{30}}{72}$

Table 1: *The Gauss integration points in* $\hat{K} := [0,1]$

over the reference element $\hat{K} := [0,1]$ using n Gauss quadrature points; see Table 1. The parameter `n` is an integer equal to 1, 2, 3 or 4, and `f` is a Matlab function handle (consult the Matlab user manual). Test your code for different choices of f. Recall that Gauss quadrature is *exact* when f is a polynomial of degree $2n-1$ or less.

(b) We partition the domain $\Omega = (0,1)$ into $N+1$ intervals $K_j = (x_{j-1}, x_j)$ with nodes $0 < x_1 < x_2 < \cdots < x_N < 1$. Consider the finite element space X_h^1 on this mesh with basis $\varphi_1, \ldots, \varphi_N$ consisting of "witch hat" functions. Write a Matlab function

```
function [A,b] = stiffness(x, f)
```

that assembles the stiffness matrix A and the load vector b,

$$A_{i,j} = \int_\Omega \varphi_i'(x)\varphi_j'(x)\,dx, \qquad b_i = \int_\Omega f(x)\varphi_i(x)\,dx.$$

(c) Collect the above pieces into a Matlab function

```
function poisson1d(x, f)
```

which approximates (3) on the given mesh with the given f, and plots the resulting solution. You can test your code with the following data:

i. $f_1(x) = 2$
ii. $f_2(x) = 6x - 2$
iii. $f_3(x) = -9\pi^2 \sin(3\pi x)$

For the above functions f it's easy to calculate the exact analytical solution of problem (3). You can use this to check if your code is working. You can use a uniform mesh in $(0,1]$, for instance, `x = linspace(0, 1, N+2)` where `N` is the number of nodes $x_1, x_2, \ldots, x_N \in (0,1)$ you want to use. Try starting with few points (2, 3, ...) and see how the solution improves adding more points (20, 50, 100, ...).