Bake, shake or break - and other applications for the FEM

Some rought theoretical background for the programming project in TMA4220 - part 2
by Kjetil André Johannesen (slightly modified by Anne Kværnø)

1 The heat equation

The heat equation reads

$$\begin{aligned} \frac{\partial u}{\partial t} &= \alpha \nabla^2 u \\ u(t,x,y,z)|_{\partial\Omega} &= u^D \\ u(t,x,y,z)|_{t=0} &= u_0(x,y,z) \end{aligned} \tag{1}$$

where α is an positive constant defined by

$$\alpha = \frac{\kappa}{c_p \rho}$$

with κ being the thermal conductivity, ρ the mass density and c_p the specific heat capacity of the material.

We are going to *semidiscretize* the system by projecting the spatial variables to a finite element subspace X_h. Multiply (1) by a test function v and integrate over the domain Ω to get

$$\iiint_\Omega \frac{\partial u}{\partial t} v \, dV = -\iiint_\Omega \alpha \nabla u \nabla v \, dV$$

Note that we have only semidiscretized the system, and as such our unknown u is given as a linear combination of the *spatial* basis functions, and continuous in time, i.e.

$$u_h(x,y,z,t) = \sum_{i=1}^{n} u_h^i(t) \varphi_i(x,y,z).$$

The variational form of the problem then reads: Find $u_h \in X_h^D$ such that

$$\begin{aligned} & \iiint_\Omega \tfrac{\partial u}{\partial t} v \, dV = -\iiint_\Omega \alpha \nabla u \nabla v \, dV, \quad \forall v \in X_h \\ \Rightarrow \quad & \sum_i \iiint_\Omega \varphi_i \varphi_j dV \tfrac{\partial u_h^i}{\partial t} = -\sum_i \iiint_\Omega \alpha \nabla \varphi_i \nabla \varphi_j dV \, u_h^i \quad \forall j \end{aligned}$$

which in turn can be written as the linear system

$$\boldsymbol{M} \frac{\partial \boldsymbol{u}}{\partial t}(t) = -\boldsymbol{A}\boldsymbol{u}(t) \tag{2}$$

which is an ordinary differential equation (ODE) with the matrices defined as

$$\boldsymbol{A} = [A_{ij}] = \iiint_\Omega \alpha \nabla \varphi_i \nabla \varphi_j \, dV$$

$$\boldsymbol{M} = [M_{ij}] = \iiint_\Omega \varphi_i \varphi_j \, dV.$$

b) Time integration

The system (2) is an ODE, which should be familiar from previous courses. Very briefly an ODE is an equation on the form

$$\frac{\partial y}{\partial t} = f(t, y)$$

where y may be a vector. The simplest ODE solver available is Eulers method

$$y_{n+1} = y_n + hf(t_n, y_n).$$

More sophisticated include the improved eulers methods

$$y_{n+1} = y_n + \frac{h}{2}\left(f(t_n, y_n) + f(t_{n+1}, y_n + hf(t_n, y_n))\right)$$

or the implicit trapezoid rule

$$y_{n+1} = y_n + \frac{h}{2}\left(f(t_n, y_n) + f(t_{n+1}, y_{n+1})\right)$$

In practise, all sufficiently stable Runge-Kutta methods or linear multistep methods can be used.

2 Structural analysis (break)

We are in this problem going to consider the linear elasticity equation. The equations describe deformation and motion in a continuum. While the entire theory of continuum mechanics is an entire course by itself, it will here be sufficient to only study a small part of this: the linear elasticity. This is governed by three main variables $\boldsymbol{u}, \boldsymbol{\varepsilon}$ and $\boldsymbol{\sigma}$ (see table 1). We will herein describe all equations and theory in terms of two spatial variables (x, y), but the extension into 3D space should be straightforward.

$\boldsymbol{u} = \begin{bmatrix} u_x \\ u_y \end{bmatrix}$	-	the *displacement* vector measures how much each spatial point has moved in (x, y)-direction
$\boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} \\ \varepsilon_{xy} & \varepsilon_{yy} \end{bmatrix}$	-	the *strain* tensor measures how much each spatial point has deformed or stretched
$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{xy} & \sigma_{yy} \end{bmatrix}$	-	the *stress* tensor measures how much forces per area are acting on a particular spatial point

Table 1: Linear elasticity variables in two dimensions

Note that the subscript denotes vector component and *not* derivative, i.e. $u_x \neq \frac{\partial u}{\partial x}$.

These three variables can be expressed in terms of each other in the following way:

$$\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x}) \quad (3)$$

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}(\boldsymbol{u}) \quad (4)$$

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}(\boldsymbol{\varepsilon}) \quad (5)$$

The primary unknown $\boldsymbol{u}$ (the displacement) is the one we are going to find in our finite element implementation. From (3) we will have two displacement values for each finite element "node", one in each of the spatial directions.

The relation (4) is a purely geometric one. Consider an infinitesimal small square of size dx and dy, and its deformed geometry as depicted in figure 1. The strain is defined as the stretching of the element, i.e. $\varepsilon_{xx} = \frac{length(ab) - length(AB)}{length(AB)}$. The complete derivations of these quantities is

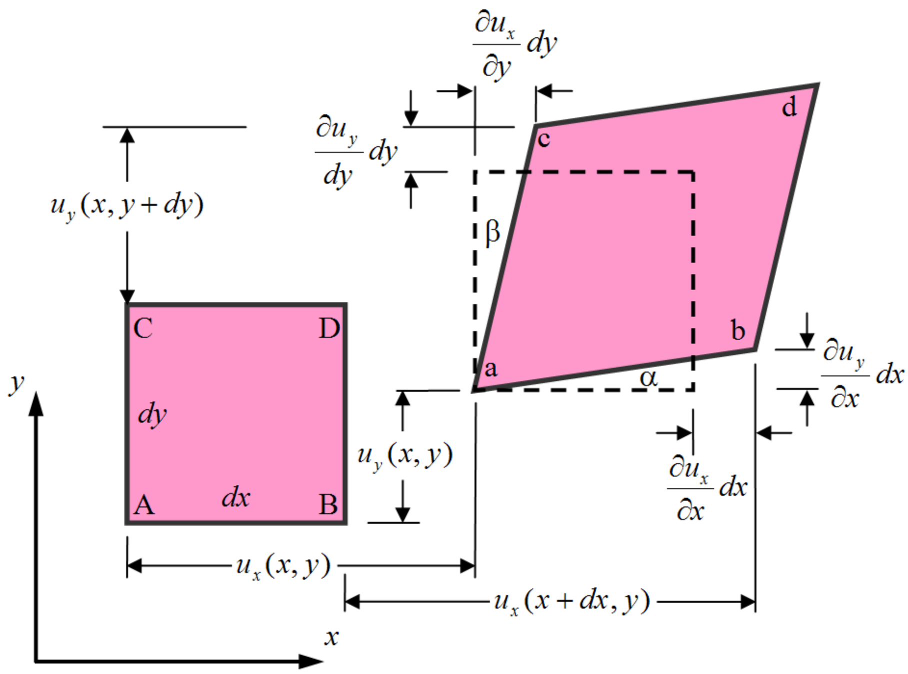

Figure 1: An infinitesimal small deformed rectangle

described well in the Wikipedia article on strain, and the result is the following relations

$$
\begin{aligned}
\varepsilon_{xx}(\boldsymbol{u}) &= \frac{\partial u_x}{\partial x} \\
\varepsilon_{yy}(\boldsymbol{u}) &= \frac{\partial u_y}{\partial y} \\
\varepsilon_{xy}(\boldsymbol{u}) &= \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right).
\end{aligned} \tag{6}
$$

Note that these relations are the *linearized* quantities, which will only be true for small deformations.

For the final relation, which connects the deformation to the forces acting upon it, we turn to the material properties. Again, there is a rich literature on the subject, and different relations or physical laws to describe different materials. In our case, we will study small deformations on solid materials like metal, wood or concrete. It is observed that such materials behave elastically when under stress of a certain limit, i.e. a deformed geometry will return to its initial state if all external forces are removed. Experiment has shown that the Generalized Hooks Law is proving remarkable accurate under such conditions. It states the following. Consider a body being dragged to each side by some stress σ_{xx} as depicted in figure 2. Hooks law states that the forces on a

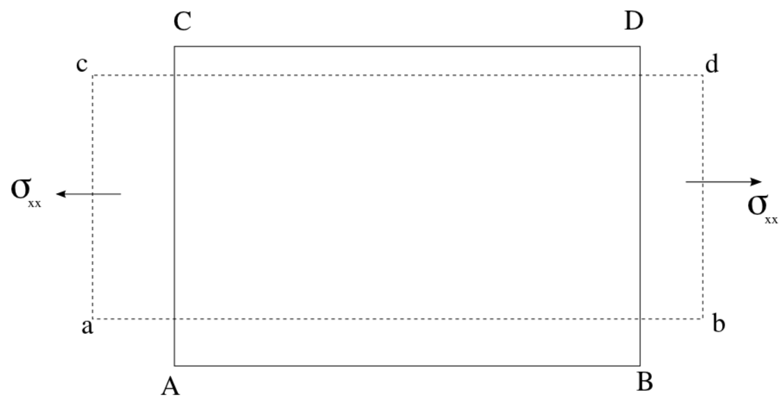

Figure 2: Deformed geometry under axial stresses

spring is linearly dependant on the amount of stretching multiplied by some stiffness constant, i.e. $\sigma_{xx} = E\varepsilon_{xx}$. The constant E is called Young's modulus. Generalizing upon this law, we see that materials typically contract in the y-direction, while being dragged in the x-direction. The ratio of compression vs expansion is called Poisson's ratio ν and is expressed as $\varepsilon_{yy} = -\nu\varepsilon_{xx}$. This gives the following relations

$$\begin{aligned} \varepsilon_{xx} &= \frac{1}{E}\sigma_{xx} \\ \varepsilon_{yy} &= -\frac{\nu}{E}\sigma_{xx} \end{aligned}$$

Due to symmetry conditions, we clearly see that when applying a stress σ_{yy} in addition to σ_{xx} we get

$$\begin{aligned} \varepsilon_{xx} &= \frac{1}{E}\sigma_{xx} - \frac{\nu}{E}\sigma_{yy} \\ \varepsilon_{yy} &= \frac{1}{E}\sigma_{yy} - \frac{\nu}{E}\sigma_{xx} \end{aligned}$$

Finally, it can be shown (but we will not) that the relation between the shear strain and shear stress is $\varepsilon_{xy} = \frac{1+\nu}{E}\sigma_{xy}$. Collecting the components of $\boldsymbol{\varepsilon}$ and $\boldsymbol{\sigma}$ in a vector, gives us the compact notation

$$\begin{aligned} \bar{\varepsilon} &= C^{-1}\bar{\sigma} \\ \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ 2\varepsilon_{xy} \end{bmatrix} &= \begin{bmatrix} \frac{1}{E} & -\frac{\nu}{E} & 0 \\ -\frac{\nu}{E} & \frac{1}{E} & 0 \\ 0 & 0 & 2\frac{1+\nu}{E} \end{bmatrix} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} \end{aligned}$$

or conversely

$$\begin{aligned} \bar{\sigma} &= C\bar{\varepsilon} \quad (7) \\ \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} &= \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{xy} \end{bmatrix} \end{aligned}$$

For a body at static equilibrium, we have the governing equations

$$\begin{aligned} \nabla\boldsymbol{\sigma}(\boldsymbol{u}) &= -\boldsymbol{f} \quad (8) \\ \left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right] \begin{bmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{xy} & \sigma_{yy} \end{bmatrix} &= -[f_x, f_y] \end{aligned}$$

and some appropriate boundary conditions

$$\boldsymbol{u} = \boldsymbol{g}, \quad on \quad \partial\Omega_D \quad (9)$$

$$\boldsymbol{\sigma}\cdot\hat{\boldsymbol{n}} = \boldsymbol{h}, \quad on \quad \partial\Omega_N \quad (10)$$

a) Weak form

It can be shown that (8) can be written as the scalar equation

$$\sum_{i=1}^{2}\sum_{j=i}^{2}\int_\Omega \varepsilon_{ij}(\boldsymbol{v})\sigma_{ij}(\boldsymbol{u})\,dA = \sum_{i=1}^{2}\int_\Omega v_i f_i\,dA + \sum_{i=1}^{2}\sum_{j=1}^{2}\int_{\partial\Omega} v_i\sigma_{ij}\hat{\boldsymbol{n}}\,d\boldsymbol{S}$$

(where we have exchanged the subscripts (x, y) with $(1, 2)$) by multiplying with a test function $\boldsymbol{v} = \begin{bmatrix} v_1(x,y) \\ v_2(x,y) \end{bmatrix}$ and integrating over the domain Ω Moreover, show that this can be written in compact vector form as

$$\begin{aligned} \int_\Omega \bar{\varepsilon}(\boldsymbol{v})^T C\bar{\varepsilon}(\boldsymbol{u})\,dA &= \int_\Omega \boldsymbol{v}^T\boldsymbol{f}\,dA + \int_{\partial\Omega} \boldsymbol{v}^T\boldsymbol{\sigma}\hat{\boldsymbol{n}}\,d\boldsymbol{S} \\ &= \int_\Omega \boldsymbol{v}^T\boldsymbol{f}\,dA + \int_{\partial\Omega} \boldsymbol{v}^T\boldsymbol{h}\,d\boldsymbol{S} \end{aligned}$$

b) Galerkin projection

As in 2b) let $\boldsymbol{v}$ be a test function in the space X_h of piecewise linear functions on some triangulation T. Note that unlike before, we now have *vector* test functions. This means that for each node $\hat{i}$, we will have two test functions

$$\begin{aligned} \boldsymbol{\varphi}_{\hat{i},1}(\boldsymbol{x}) &= \begin{bmatrix} \varphi_{\hat{i}}(\boldsymbol{x}) \\ 0 \end{bmatrix} \\ \boldsymbol{\varphi}_{\hat{i},2}(\boldsymbol{x}) &= \begin{bmatrix} 0 \\ \varphi_{\hat{i}}(\boldsymbol{x}) \end{bmatrix} \end{aligned}$$

Let these functions be numbered by a single running index $i = 2\hat{i} + d$, where i is the node number in the triangulation and d is the vector component of the function.

Show that by inserting $\boldsymbol{v} = \boldsymbol{\varphi}_j$ and $\boldsymbol{u} = \sum_i \boldsymbol{\varphi}_i u_i$ into (11) you get the system of linear equations

$$A\boldsymbol{u} = \boldsymbol{b}$$

where

$$\begin{aligned} A = [A_{ij}] &= \int_\Omega \bar{\varepsilon}(\boldsymbol{\varphi}_i)^T C \bar{\varepsilon}(\boldsymbol{\varphi}_j),\ dA \\ b = [b_i] &= \int_\Omega \boldsymbol{\varphi}_i^T \boldsymbol{f}\, dA + \int_{\partial\Omega} \boldsymbol{\varphi}_i^T \boldsymbol{h}\, d\boldsymbol{S} \end{aligned}$$

(**Hint:** $\bar{\varepsilon}(\cdot)$ is a linear operator)

Stress analysis

Solving (8) with a finite element method gives you the primary unknown: the displacement u. If you are interested in derived quantities such as the stresses, these can be calculated from (7). Note that σ is in essence the derivative of u which means that since u is C^0 across element boundaries, then σ will be discontinuous. To get stresses at the nodal values, we propose to average the stresses over all neighbouring elements.

Loop over all elements and evaluate (the constant) stresses on that element. For each node, assign the stresses to be the average stress over all neighbouring elements. This method is called "Stress Recovery".

3 Vibration analysis (shake)

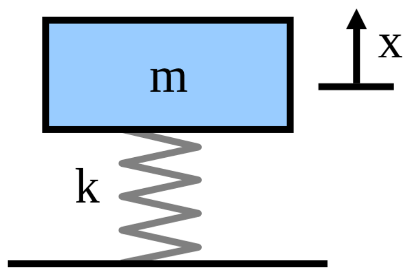

Figure 3: Mass-spring-model

The forces acting on a point mass m by a spring is given by the well known Hooks law:

$$m\ddot{x} = -kx$$

This can be extended to multiple springs and multiple bodies as in figure 4

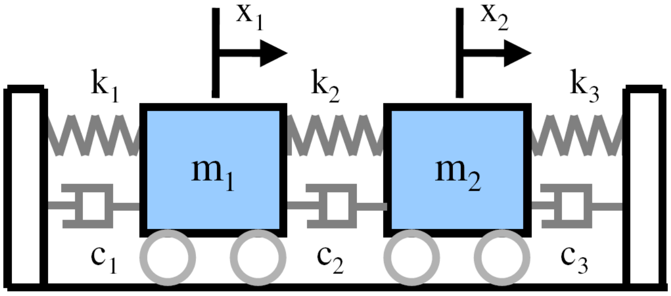

Figure 4: 2 degree-of-freedom mass spring model

The physical laws will now become a system of equations instead of the scalar one above. The forces acting on m_1 is the spring k_1 dragging in negative direction and k_2 dragging in the positive direction.

$$m_1\ddot{x_1} = -k_1 x_1 + k_2(x_2 - x_1)$$

This is symmetric, and we have an analogue expression for m_2. The system can be written in matrix form as

$$\begin{aligned} \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{bmatrix} \ddot{x_1} \\ x_2 \end{bmatrix} &= \begin{bmatrix} -k_1 - k_2 & k_2 \\ k_2 & -k_2 - k_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ M\ddot{\boldsymbol{x}} &= A\boldsymbol{x} \end{aligned}$$

When doing continuum mechanics, it is the exact same idea, but the actual equations differ some. Instead of discrete equations, we have continuous functions in space and the governing equations are

$$\rho\ddot{\boldsymbol{u}} = \nabla\boldsymbol{\sigma}(\boldsymbol{u})$$

semi-discretization yields the following system of equations

$$M\ddot{\boldsymbol{u}} = -A\boldsymbol{u} \tag{11}$$

with the usual stiffness and mass matrix

$$\begin{aligned} \boldsymbol{A} = [A_{ij}] &= \iiint_\Omega \bar{\varepsilon}(\boldsymbol{\varphi}_i)^T C \bar{\varepsilon}(\boldsymbol{\varphi}_j)\, dV \\ \boldsymbol{M} = [M_{ij}] &= \iiint_\Omega \rho \boldsymbol{\varphi}_i^T \boldsymbol{\varphi}_j\, dV. \end{aligned}$$

We are now going to search for solutions of the type:

$$\boldsymbol{u} = \boldsymbol{u}e^{\omega i t} \tag{12}$$

which inserted into (11) yields

$$\omega^2 M\boldsymbol{u} = A\boldsymbol{u} \tag{13}$$

This is called the generalized eigenvalue problem.