Norwegian University of Science and Technology
Department of Mathematical Sciences

# TMA4220 Numerical Solution of Partial Differential Equations Using Element Methods
Høst 2014

**Exercise set 4**

$\boxed{1}$ Let $a(u, v) = \int_\Omega \nabla u \nabla v d\Omega$.

**a)** Find the element stiffness matrix $A_h^K \in \mathbb{R}^{3\times3}$ for an element with nodes (vertices) $x_1^K = (0, 0)$, $x_2^K = (h, 0)$ and $x_3^K(0, h)$.

**b)** Prove that your results applies to *any* right-triangular element with hypotenuse $\sqrt{2}h$ at *any* orientation or position. **Hint:** First prove it applies to a right-triangular element with hypotenuse $\sqrt{2}h$ and the right-angle corner in $(0, 0)$.

**c)** The mass matrix $M_h$ is given by $(M_h)_{i,j} = \int_\Omega \varphi_j \varphi_i d\Omega$. Find the element mass matrix $M_h^K$ for any right-triangular element with hypotenuse $\sqrt{2}h$.

$\boxed{2}$ From the exam set 2006.

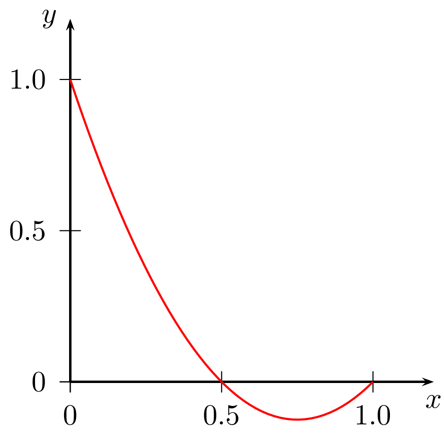


Figure 1: A one-dimensional quadratic element

Figure 1 shows a quadratic element with domain $\Omega_{T_h^{1D}} = (0, 1)$. The element has 3 local nodes: $x_1 = 0$, $x_2 = 0.5$ and $x_3 = 1$. The shape function $\psi_1^{1D}$ for local node 1, presented in the figure, is given by

$$\psi_1^{1D}(x) = 2\big(x - \frac{1}{2}\big)\big(x - 1\big).$$

**a)** Copy the figure to your own answer sheet, and draw the shape functions corresponding to node 2 and 3. Write down the shape functions as functions of $x$.

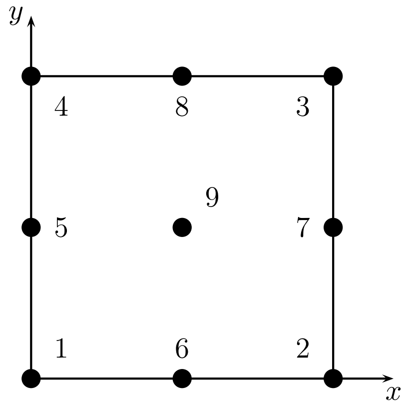


Figure 2: A two-dimensional quadratic element

Figure 2 shows the domain $\Omega = (0,1) \times (0,1)$ discretized with one quadratic finite element. The element has 9 local nodes, each labelled as shown.

**b)** Write down the 9 quadratic basis functions as functions of $x$ and $y$.
*Hint: Use the basis functions from the one-dimensional example in point* ***a)****.*

Consider now the Poisson problem

$$\begin{aligned} -\Delta u &= 1 \qquad \text{in } \Omega = (0,1) \times (0,1) \\ u &= 0 \qquad \text{on } \partial\Omega \end{aligned} \tag{1}$$

**c)** Derive the weak formulation of the problem, that is

$$\text{Find } u \in X \text{ such that} \qquad a(u,v) = l(v), \qquad \forall v \in X.$$

Determine the expressions of the function space $X$ and the forms $a(\cdot,\cdot)$ and $l(\cdot)$.

We now discretize the Poisson problem (1) from point **c)** by one quadratic element as depicted in Figure 2. We express the the solution $u_h$ by help of the basis functions, insert it into the weak formulation from point **c)** and includes the boundary conditions. This will give the linear system

$$\mathbf{A}_h u_h = \mathbf{F}_h, \tag{2}$$

where $(A_h)_{i,j} = a(\psi_j, \psi_i)$ and $(\mathbf{F}_h)_i = l(\psi_i)$.

**d)** How many degrees of freedoms has the linear system of equations?

Parts of the local stiffness matrix for the quadratic 9-nodal element in point **b)** is given below.

$$A_h^{T_h^{2D}} = \begin{bmatrix} X & \frac{-1}{30} & \frac{-1}{45} & \frac{-1}{30} & \frac{-1}{5} & \frac{-1}{5} & \frac{1}{9} & \frac{1}{9} & \frac{-16}{45} \\ \frac{-1}{30} & \frac{28}{45} & \frac{-1}{30} & \frac{-1}{45} & -\frac{1}{9} & \frac{-1}{5} & \frac{-1}{5} & \frac{1}{9} & \frac{-16}{45} \\ \frac{-1}{45} & \frac{-1}{30} & \frac{28}{45} & \frac{-1}{30} & \frac{1}{9} & \frac{1}{9} & \frac{-1}{5} & \frac{-1}{5} & \frac{-16}{45} \\ \frac{-1}{30} & \frac{-1}{45} & \frac{-1}{30} & \frac{28}{45} & \frac{-1}{5} & \frac{1}{9} & \frac{1}{9} & \frac{-1}{5} & \frac{-16}{45} \\ \frac{-1}{5} & \frac{1}{9} & X & \frac{-1}{5} & \frac{88}{45} & \frac{-16}{45} & 0 & \frac{-16}{45} & \frac{-16}{15} \\ \frac{-1}{5} & \frac{-1}{5} & \frac{1}{9} & \frac{1}{9} & \frac{-16}{45} & \frac{88}{45} & \frac{-16}{45} & 0 & \frac{-16}{15} \\ \frac{1}{9} & \frac{-1}{5} & \frac{-1}{5} & \frac{1}{9} & 0 & \frac{-16}{45} & \frac{88}{45} & \frac{-16}{45} & \frac{-16}{15} \\ \frac{1}{9} & \frac{1}{9} & \frac{-1}{5} & \frac{-1}{5} & \frac{-16}{45} & 0 & \frac{-16}{45} & \frac{88}{45} & \frac{-16}{15} \\ \frac{-16}{45} & \frac{-16}{45} & \frac{-16}{45} & \frac{-16}{45} & \frac{-16}{15} & \frac{-16}{15} & \frac{-16}{15} & \frac{-16}{15} & \frac{256}{45} \end{bmatrix}$$

**e)** The elements $(A_h^{T_h^{2D}})_{1,1}$ and $(A_h^{T_h^{2D}})_{5,3}$ are missing. Find them.

The local load vector is given by

$$\mathbf{F}_h = \begin{bmatrix} \frac{1}{36} & \frac{1}{36} & \frac{1}{36} & \frac{1}{36} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{4}{9} \end{bmatrix}^T$$

**f)** Solve the linear system (2) and find the numerical solution $u_h(x, y)$.

$\boxed{3}$ **a)** Prove that the finite element depicted in Figure 2 is a finite element, which forms a $C^0$ finite element space.

**b)** Show that the finite element space constructed from triangular cubic Hermite elements (p. 75 in B&S) is $C^0$ but not $C^1$.