Norwegian University of Science and Technology
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TMA4215
Numerical Mathematics
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Exercise set 7:
Suggested solutions

Sol:1 **a)** We have

$$\text{Exact solution:} \qquad y(t_{n+1}) = y(t_n) + h\Phi\big(t_n, y(t_n); h\big) + d_{n+1},$$
$$\text{Numerical solution:} \qquad y_{n+1} = y_n + h\Phi(t_n, y_n; h).$$

Take the difference between these, and use the fact that the global error in step n is given by $e_n = y(t_n) - y_n$:

$$e_{n+1} = e_n + h\Big(\Phi\big(t_n, y(t_n); h\big) - \Phi(t_n, y_n; h)\Big) + d_{n+1}.$$

Take the norm of both sides and apply the triangle inequality for norms:

$$\begin{aligned}\|e_{n+1}\| &= \Big\|e_n + h\Big(\Phi\big(t_n, y(t_n); h\big) - \Phi(t_n, y_n; h)\Big) + d_{n+1}\Big\| \\ &\leq \|e_n\| + h\big\|\Phi\big(t_n, y(t_n); h\big) - \Phi(t_n, y_n; h)\big\| + \|d_{n+1}\|.\end{aligned}$$

Use the assumptions given in the text:

$$\|e_{n+1}\| \leq \|e_n\| + hM\|y(t_n) - y_n\| + Dh^{p+1} = (1 + hM)\|e_n\| + Dh^{p+1}.$$

Assuming that $y(t_0) = y_0$, we now have

$$\begin{aligned}\|e_1\| &\leq Dh^{p+1}, \\ \|e_2\| &\leq (1 + hM)Dh^{p+1} + Dh^{p+1}, \\ &\vdots \\ \|e_n\| &\leq Dh^{p+1}\sum_{i=0}^{n-1}(1 + hM)^i, \\ &\vdots\end{aligned}$$

so that

$$\|e_N\| \leq Dh^{p+1}\sum_{i=0}^{N-1}(1 + hM)^i = \frac{(1 + hM)^N - 1}{1 + hM - 1}Dh^{p+1} = \frac{(1 + hM)^N - 1}{M}Dh^p.$$

Utilise the fact that $\mathrm{e}^x \geq 1 + x$ for $x > 0$, so that

$$\|e_N\| \leq \frac{\mathrm{e}^{MhN} - 1}{M}Dh^p = \frac{\mathrm{e}^{M(t_{\text{end}} - t_0)} - 1}{M}Dh^p = Ch^p.$$

This argument does not depend on the choice of norm.

b) We know that f satisfies a Lipschitz condition in y, i.e.

$$\|f(t,y) - f(t,\tilde{y})\| \leq L\|y - \tilde{y}\|.$$

The increment function $\Phi(t_n, y_n; h)$ for the given method is

$$\Phi(t_n, y_n; h) = b_1 k_1 + b_2 k_2.$$

Let us calculate this for two different starting values y_n and $\tilde{y}_n$.

$$\begin{aligned} k_1 &= f(t_n, y_n), & \tilde{k}_1 &= f(t_n, \tilde{y}_n), \\ k_2 &= f(t_n + c_2 h, y_n + h c_2 k_1), & \tilde{k}_2 &= f(t_n + c_2 h, \tilde{y}_n + h c_2 \tilde{k}_1). \end{aligned}$$

Thus,

$$\Phi(t_n, y_n; h) - \Phi(t_n, \tilde{y}_n; h) = b_1(k_1 - \tilde{k}_1) + b_2(k_2 - \tilde{k}_2).$$

Take the norm of both sides and use the properties of norms as well as Lipschitz continuity for f:

$$\begin{aligned} \|\Phi(t_n, y_n; h) - \Phi(t_n, \tilde{y}_n; h))\| &= \|b_1(k_1 - \tilde{k}_1) + b_2(k_2 - \tilde{k}_2)\| \\ &\leq |b_1|\|k_1 - \tilde{k}_1\| + |b_2|\|k_2 - \tilde{k}_2\| \\ &\leq |b_1|L\|y_n - \tilde{y_n}\| + |b_2|L\|y_n + hc_2k_1 - \tilde{y}_n - hc_2\tilde{k}_1\| \\ &\leq |b_1|L\|y_n - \tilde{y_n}\| + |b_2|L(\|y_n - \tilde{y}_n\| + h|c_2|L\|y_n - \tilde{y}_n\|) \end{aligned}$$

Using that $h \leq h_{\max}$, we have now shown that

$$\|\Phi(t_n, y_n; h) - \Phi(t_n, \tilde{y}_n; h))\| \leq M\|y_n - \tilde{y}_n\|$$

with

$$M = L(|b_1| + |b_2|) + h_{\max} L^2 |b_2 c_2|.$$

Sol:2 **a)** The eight order conditions for fourth order Runge–Kutta methods are:

$$\sum_i b_i = 1 \tag{1}$$

$$\sum_i b_i c_i = \frac{1}{2} \tag{2}$$

$$\sum_i b_i c_i^2 = \frac{1}{3} \tag{3}$$

$$\sum_{i,j} b_i a_{ij} c_j = \frac{1}{6} \tag{4}$$

$$\sum_i b_i c_i^3 = \frac{1}{4} \tag{5}$$

$$\sum_{i,j} b_i c_i a_{ij} c_j = \frac{1}{8} \tag{6}$$

$$\sum_{i,j} b_i a_{ij} c_j^2 = \frac{1}{12} \tag{7}$$

$$\sum_{i,j,k} b_i a_{ij} a_{jk} c_k = \frac{1}{24} \tag{8}$$

To check that the given method satisfies the order conditions, we simply insert the values from the Butcher tableau. E.g. for (7), we get

$$\begin{aligned}\sum_{i,j} b_i a_{ij} c_j^2 &= b_3 a_{32} c_2^2 + b_4(a_{42}c_2^2 + a_{43}c_3^2)\\ &= \frac{1}{3}\frac{1}{2}\left(\frac{1}{2}\right)^2 + \frac{1}{6}\left(0\cdot\left(\frac{1}{2}\right)^2 + 1\cdot\left(\frac{1}{2}\right)^2\right) = \frac{1}{12}.\end{aligned}$$

b) We want to find a set of weights $\hat{b}_i$ so that the conditions (1)–(4) are satisfied. This means that

$$\begin{aligned}\hat{b}_1 + \hat{b}_2 + \hat{b}_3 + \hat{b}_4 &= 1,\\ \frac{1}{2}\hat{b}_2 + \frac{1}{2}\hat{b}_3 + \hat{b}_4 &= \frac{1}{2},\\ \frac{1}{4}\hat{b}_2 + \frac{1}{4}\hat{b}_3 + \hat{b}_4 &= \frac{1}{3},\\ \frac{1}{4}\hat{b}_3 + \frac{1}{2}\hat{b}_4 &= \frac{1}{6}.\end{aligned}$$

This system has a unique solution, namely the weights of the original Kutta's method, $\hat{b}_1 = \hat{b}_4 = \frac{1}{6}$ and $\hat{b}_2 = \hat{b}_3 = \frac{1}{3}$. Thus, we can not use this solution for comparison.

Sol:3 **a)** The eigenvalues are $\lambda_{1,2} = -10 \pm 20\mathrm{i}$.

b) The stability function is given by

$$R(z) = 1 + z + \frac{1}{2}z^2.$$

We must have

$$|R(h\lambda)| \leq 1$$

for the numerical solution to be stable for all eigenvalues of M. In our case, this means that

$$\begin{aligned}\big|R\big((-10+20\mathrm{i})h\big)\big|^2 &= R\big((-10+20\mathrm{i})h\big)\cdot R\big((-10-20\mathrm{i})h\big)\\ &= 1 - 20h + 200h^2 - 5000h^3 + 62500h^4 \leq 1,\end{aligned}$$

which is satisfied when $0 \leq h \leq 0.08603$.

c) See `impEuler.py`. Use for instance the step lengths 0.085, 0.086 and 0.087.