Norwegian University of Science and Technology
Department of Mathematical Sciences

TMA4215 Numerical Mathematics Autumn 2017

Exercise set 6: Suggested solutions

Sol:1 We study Hermite interpolation, which is characterised by a polynomial $p(x)$ defined on $n+1$ distinct nodes $x_0, x_1, \dots, x_n$ satisfying the conditions

$$p(x_i) = y_i, \quad p'(x_i) = v_i, \quad i = 0, 1, \dots, n \tag{1}$$

where $\{y_i\}_{i=0}^n$ and $\{v_i\}_{i=0}^n$ are arbitrary, specified values.

a) It is reasonable to assume $p \in \mathbb{P}_{2n+1}$ since (1) specifies $2n+2$ conditions (2 conditions for each of the $n+1$ points). A polynomial of degree $2n+1$ can generally be represented by

$$a_0 + a_1 x + a_2 x^2 + \cdots + a_{2n} x^{2n} + a_{2n+1} x^{2n+1}$$

and thus has $2n+2$ parameters $a_0, a_1, \dots, a_{2n+1}$. Hence, we can use the conditions (1) to uniquely determine these parameters.

b) We assume that the functions $A_i(x)$ and $B_i(x)$ (which are not specified for now), all defined for $i = 0, 1, \dots, n$, satisfy

$$\begin{aligned} A_i(x_j) &= \delta_{ij}, & B_i(x_j) &= 0, \\ A_i'(x_j) &= 0, & B_i'(x_j) &= \delta_{ij} \end{aligned} \tag{2}$$

for all $i, j = 0, 1, \dots, n$. We define the function $g(x)$ as

$$g(x) = \sum_{i=0}^n y_i A_i(x) + \sum_{i=0}^n v_i B_i(x). \tag{3}$$

Note: We have not yet said *anything* about which type of function $A_i(x)$ and $B_i(x)$ are, just that they shall satisfy the conditions (2).

Given the function $g(x)$ in (3) we find for arbitrary $j = 0, 1, \dots, n$

$$\begin{aligned} g(x_j) &= \sum_{i=0}^n y_i A_i(x_j) + \sum_{i=0}^n v_i B_i(x_j) = \sum_{i=0}^n y_i \delta_{ij} + \sum_{i=0}^n v_i \cdot 0 = y_j, \\ g'(x_j) &= \sum_{i=0}^n y_i A_i'(x_j) + \sum_{i=0}^n v_i B_i'(x_j) = \sum_{i=0}^n y_i \cdot 0 + \sum_{i=0}^n v_i \delta_{ij} = v_j. \end{aligned}$$

Thus, we have shown that a function $g(x)$ as defined in (3) satisfies (1) as long as the *basis functions* $A_i(x)$ and $B_i(x)$ satisfy (2).

c) We will now look at possible representations of the basis functions $A_i(x)$ and $B_i(x)$. Specifically, we look at basis functions $A_i \in \mathbb{P}_{2n+1}$ and $B_i \in \mathbb{P}_{2n+1}$. We use the cardinal functions

$$L_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^n \frac{x - x_j}{x_i - x_j}$$

which satisfy $L_i(x_j) = \delta_{ij}$ and study the functions

$$A_i(x) = \big(1 - 2(x - x_i)L_i'(x_i)\big)L_i^2(x), \quad B_i(x) = (x - x_i)L_i^2(x). \tag{4}$$

It is obvious that $A_i \in \mathbb{P}_{2n+1}$ and $B_i \in \mathbb{P}_{2n+1}$. Next, we find that

$$\begin{aligned} A_i'(x) &= -2L_i'(x_i)L_i^2(x) + 2\big(1 - 2(x - x_i)L_i'(x_i)\big)L_i(x)L_i'(x) \\ &= 2L_i(x)L_i'(x)\big(1 - L_i(x) - 2(x - x_i)L_i'(x_i)\big) \\ B_i'(x) &= L_i^2(x) + 2(x - x_i)L_i(x)L_i'(x) = L_i(x)\big(L_i(x) + 2(x - x_i)L_i'(x)\big). \end{aligned}$$

We see that the polynomials $A_i(x)$ and $B_i(x)$ satisfy (2) for all $i, j = 0, 1, \ldots, n$. Thus, the polynomials (4) are polynomials of degree $2n + 1$ which satisfy the necessary basis conditions and can be used as basis functions for construction of *polynomials* of degree $2n + 1$ satisfying conditions (1).

d) We shall find a third degree polynomial $p(x)$ satisfying

$$p(1) = 1, \quad p'(1) = 3, \quad p(2) = 14, \quad p'(2) = 24.$$

In this case $2n + 1 = 3 \Rightarrow n = 1$ and we find

$$\begin{aligned} L_0(x) &= \frac{x-2}{1-2} = 2 - x, & L_1(x) &= \frac{x-1}{2-1} = x - 1 \\ L_0'(x) &= -1, & L_1'(x) &= 1 \\ L_0^2(x) &= x^2 - 4x + 4, & L_1^2(x) &= x^2 - 2x + 1 \end{aligned}$$

From the representation (4) we find

$$\begin{aligned} A_0(x) &= \big(1 - 2(x - 1)\cdot(-1)\big)(x^2 - 4x + 4) = (2x - 1)(x^2 - 4x + 4) \\ A_1(x) &= \big(1 - 2(x - 2)\cdot 1\big)(x^2 - 2x + 1) = (5 - 2x)(x^2 - 2x + 1) \\ B_0(x) &= (x - 1)(x^2 - 4x + 4) \\ B_1(x) &= (x - 2)(x^2 - 2x + 1). \end{aligned}$$

Thus, the third degree polynomial satisfying the conditions above is given by

$$\begin{aligned} p(x) &= (2x - 1)(x^2 - 4x + 4) + 3\cdot(x - 1)(x^2 - 4x + 4) \\ &\quad + 14\cdot(5 - 2x)(x^2 - 2x + 1) + 24\cdot(x - 2)(x^2 - 2x + 1) \\ &= x^3 + 6x^2 - 12x + 6. \end{aligned}$$

The final result follows easily from a little calculation.

Sol:2 Let $T_1 = T(a, b) = (b - a)\big(f(a) + f(b)\big)/2$, and $T_2 = T(a, c) + T(c, b)$, where $c = (a + b)/2$. We know that

$$\begin{aligned} \int_a^b f(x)\,\mathrm{d}x &= T_1 - \frac{(b-a)^3}{12} f''(\xi_1), \\ \int_a^b f(x)\,\mathrm{d}x &= T_2 - \frac{(b-a)^3}{12\cdot 2^3}\left(f''(\eta_1) + f''(\eta_2)\right), \end{aligned}$$

where $\xi_1 \in (a, b)$, $\eta_1 \in (a, c)$ and $\eta_2 \in (c, b)$. If we assume that $f''(x)$ changes little over the interval (a, b), we can use the same reasoning as for the adaptive Simpson's rule. An appropriate error estimate for T_2 is

$$\int_a^b f(x)\,\mathrm{d}x - T_2 \approx \mathcal{E}(a, b) = \frac{1}{3}(T_2 - T_1).$$

The adaptive trapezoid algorithm then becomes:

function ADAPTIVE-TRAPEZOID(f, a, b, tol)
  $T_1 \leftarrow T(a, b)$ $\triangleright\ T(a,b) = (b-a)\big(f(a) + f(b)\big)/2$
  $c \leftarrow (a + b)/2$
  $T_2 \leftarrow T(a, c) + T(c, b)$
  $\mathcal{E} \leftarrow (T_2 - T_1)/3$
  if $|\mathcal{E}| \le tol$ **then**
    return T_2
  else
    $T_l \leftarrow$ ADAPTIVE-TRAPEZOID$(f, a, c, tol/2)$
    $T_r \leftarrow$ ADAPTIVE-TRAPEZOID$(f, c, b, tol/2)$
    return $T_l + T_r$
  end if
end function

Applied to the integral in the text, the algorithm gives:

<table>
<tr><td colspan="4">$tol = 2 \cdot 10^{-3},\ a = 0.0,\ b = 0.8$
$T_1 = 0.23888,\ T_2 = 0.18317,\ \mathcal{E} = -1.86 \cdot 10^{-2}$</td></tr>
<tr><td colspan="2">$tol = 1 \cdot 10^{-3}\ a = 0.0,\ b = 0.4$
$T_1 = 0.031864,\ T_2 = 0.0239330$
$\mathcal{E} = -2.6 \cdot 10^{-3}$</td><td colspan="2">$tol = 1 \cdot 10^{-3}\ a = 0.4,\ b = 0.8$
$T_1 = 0.15130,\ T_2 = 0.14611$
$\mathcal{E} = -1.7 \cdot 10^{-3}$</td></tr>
<tr><td>$tol = 5 \cdot 10^{-4}$
$a = 0.0,\ b = 0.2$
$T_1 = 0.004000$
$T_2 = 0.003000$
$\mathcal{E} = -3.3 \cdot 10^{-4}$</td><td>$tol = 5 \cdot 10^{-4}$
$a = 0.2,\ b = 0.4$
$T_1 = 0.019931$
$T_2 = 0.018953$
$\mathcal{E} = -3.3 \cdot 10^{-4}$</td><td>$tol = 5 \cdot 10^{-4}$
$a = 0.4,\ b = 0.6$
$T_1 = 0.051159$
$T_2 = 0.050320$
$\mathcal{E} = -2.8 \cdot 10^{-4}$</td><td>$tol = 5 \cdot 10^{-4}$
$a = 0.6,\ b = 0.8$
$T_1 = 0.094947$
$T_2 = 0.094536$
$\mathcal{E} = -1.4 \cdot 10^{-4}$</td></tr>
</table>

So

$$T = 0.003 + 0.018953 + 0.050320 + 0.094536 = 0.1668.$$

Sol:3 **a)**

$$\int_{-1}^{1} \frac{\mathrm{e}^x}{\sqrt{1-x^2}}\,\mathrm{d}x \approx \frac{\mathrm{e}^{-1/\sqrt{3}}}{\sqrt{1-1/3}} + \frac{\mathrm{e}^{1/\sqrt{3}}}{\sqrt{1-1/3}} = \sqrt{6}\cosh(1/\sqrt{3}) \approx 2.8692$$

b) Use the inner product $\langle f, g\rangle = \int_{-1}^{1}\big(f(x)g(x)/\sqrt{1-x^2}\big)\,\mathrm{d}x$, with Chebyshev polynomials as orthogonal polynomials. We choose $n = 1$, $T_2(x) = 2x^2 - 1$, i.e. $x_0 = -1/\sqrt{2}$, $x_1 = 1/\sqrt{2}$. The weights become

$$A_0 = \int_{-1}^{1} \frac{\big(x - 1/\sqrt{2}\big)}{\big(-1/\sqrt{2} - 1/\sqrt{2}\big)} \cdot \frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x = \frac{\pi}{2},$$
$$A_1 = \int_{-1}^{1} \frac{\big(x + 1/\sqrt{2}\big)}{\big(1/\sqrt{2} + 1/\sqrt{2}\big)} \cdot \frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x = \frac{\pi}{2}.$$

The approximation becomes $\frac{\pi}{2}(\mathrm{e}^{-1/\sqrt{2}} + \mathrm{e}^{1/\sqrt{2}}) \approx 3.9603$ which is significantly better than the answer in **a)**.

c) The error is given by

$$E = K \cdot f^{(4)}(\nu), \qquad K = \frac{1}{4!}\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}\left(x^2 - \frac{1}{2}\right)^2 \mathrm{d}x = \frac{\pi}{192}.$$

Since $|\mathrm{e}^x| < \mathrm{e}$ on $(-1,1)$, we must have $|E| < \mathrm{e}\pi/192 \approx 0.044$. The measured error is 0.017.

Sol:4 The recursion formula from the note gives

$$\begin{aligned}
&\phi_0(x) = 1, && \langle \phi_0, \phi_0 \rangle = \int_0^\infty \mathrm{e}^{-x}\,\mathrm{d}x = 1 && \\
& && \langle x\phi_0, \phi_0 \rangle = \int_0^\infty \mathrm{e}^{-x} x\,\mathrm{d}x = 1 && \Rightarrow B_1 = 1 \\
&\phi_1(x) = x - 1, && \langle \phi_1, \phi_1 \rangle = \int_0^\infty \mathrm{e}^{-x}(x-1)^2\,\mathrm{d}x = 1 && \\
& && \langle x\phi_1, \phi_1 \rangle = 3 && \Rightarrow B_2 = 3, C_2 = 1 \\
&\phi_2(x) = (x-3)\phi_1(x) - \phi_0(x) = x^2 - 4x + 3 - 1 = x^2 - 4x + 2 && &&
\end{aligned}$$

Sol:5 We must show that

$$\int_{-1}^{1} \left((x^2-1)^k\right)^{(k)} \left((x^2-1)^j\right)^{(j)} \mathrm{d}x = 0 \quad \text{for all } j < k.$$

Partial integration, $\int_a^b u\,\mathrm{d}v = uv|_a^b - \int_a^b v\,\mathrm{d}u$ with

$$u = \left((x^2-1)^j\right)^{(j)}, \qquad \mathrm{d}v = \left((x^2-1)^k\right)^{(k)} \mathrm{d}x$$

applied to the integral above gives

$$\left((x^2-1)^k\right)^{(k-1)} \left((x^2-1)^j\right)^{(j)}\Big|_{-1}^{1} - \int_{-1}^{1} \left((x^2-1)^k\right)^{(k-1)} \left((x^2-1)^j\right)^{(j+1)} \mathrm{d}x.$$

The first term is zero (see hint). Again, we apply partial integration to the integral. After having done this $j+1$ times, we end up with

$$(-1)^{j+1}\int_{-1}^{1} \left((x^2-1)^k\right)^{(k-j-1)} \left((x^2-1)^j\right)^{(2j+1)} \mathrm{d}x = 0$$

since $\left((x^2-1)^j\right)^{(2j+1)} = 0$.

Sol:6 See the exam paper solution, December 2008, Problem 3