TMA4215
Numerical Mathematics
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Exercise set 5: Suggested solutions

Sol:1 **a)** The table of divided differences is ($f[x_i] = y_i$):

x_i	$f[x_i]$	$f[x_i, x_{i+1}]$	$f[x_i, x_{i+1}, x_{i+2}]$
0	1		
		1/2	
2	2		1/2
		2	
3	4		

and we end up with the polynomial

$$p_2(x) = 1 + \frac{1}{2}x + \frac{1}{2}x(x-2) = \frac{1}{2}x^2 - \frac{1}{2}x + 1.$$

b) We keep the table from **a)** and just add one more row:

x_i	$f[x_i]$	$f[x_i, x_{i+1}]$	$f[x_i, x_{i+1}, x_{i+2}]$	$f[x_i, x_{i+1}, x_{i+2}, x_{i+3}]$
0	1			
		1/2		
2	2		1/2	
		2		$-1/2$
3	4		0	
		2		
1	0			

The polynomial becomes

$$p_3(x) = 1 + \frac{1}{2}x + \frac{1}{2}x(x-2) - \frac{1}{2}x(x-2)(x-3) = -\frac{1}{2}x^3 + 3x^2 - \frac{7}{2}x + 1.$$

Sol:2 **a)** Equidistant nodes: $x_i = ih$, $i = 0, \dots, 3$ with $h = \pi/3$. We are to interpolate the points in the table

x_i	0	$\pi/3$	$2\pi/3$	π
$f(x_i)$	0	$\sqrt{3}/2$	$\sqrt{3}/2$	0

Result:

$$p_3(x) = \frac{9\sqrt{3}}{4\pi^2}(-x^2 + \pi x),$$

with error bound (see Exercise 5, Task 3)

$$|\sin(x) - p_3(x)| \leq \frac{1}{16}\left(\frac{\pi}{3}\right)^4 \approx 0.0752.$$

b) The Chebyshev nodes are: $x_i = \pi/2 + (\pi/2)\cos\big((2i+1)\pi/8\big)$ when $n = 3$. This gives us the table

x_i	3.022022903	2.171914057	0.9696785970	0.119569751
$f(x_i)$	0.1192850409	0.8247039815	0.8247039818	0.1192850413

The polynomial is

$$p_3(x) = -0.4043173324\,x^2 + 1.270200363\,x - 0.0268120051$$

with error bound

$$|\sin(x) - p_3(x)| \le \frac{1}{8 \cdot 4!}\left(\frac{\pi}{2}\right)^4 \approx 0.0317.$$

c) We have $M_n = \max_{x\in[0,\pi]}|f^{(n+1)}(x)| = 1$.
Equidistant nodes:

$$|f(x) - p_n(x)| \le \frac{1}{4(n+1)}\left(\frac{\pi}{n}\right)^{n+1}$$

Chebyshev nodes: The change of variables leads to (see Theorem 8.7 with proof in S&M)

$$\prod_{i=0}^{n}(x - x_i) = \left(\frac{b-a}{2}\right)^{n+1}\prod_{i=0}^{n}(t - t_i) \qquad \Rightarrow \qquad \prod_{i=0}^{n}|x - x_i| \le \left(\frac{b-a}{2}\right)^{n+1}\frac{1}{2^n}$$

so that the error bound becomes

$$|f(x) - p_n(x)| \le \left(\frac{\pi}{4}\right)^{n+1}\frac{2}{(n+1)!}.$$

In Figure 1 we see that Chebyshev nodes give lower error bound than equidistant nodes.

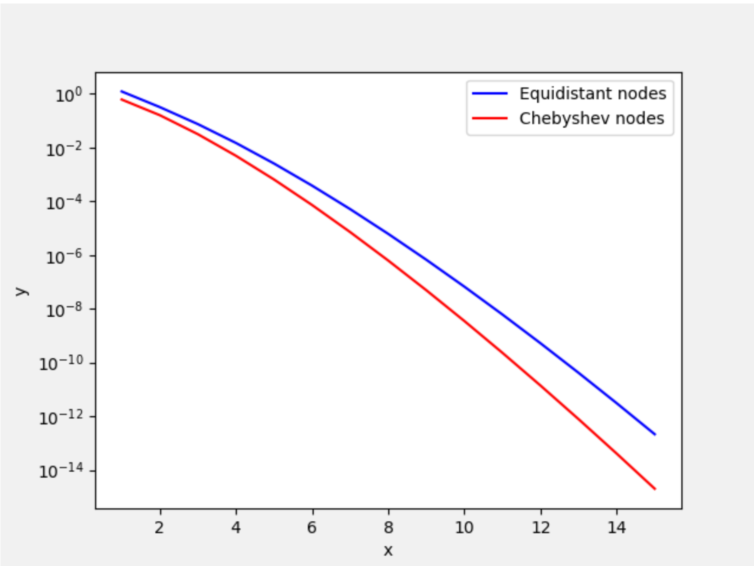

Figure 1: Error bounds for the interpolation polynomials.

Sol:3 See the provided PYTHON files.

Sol:4 **a)** We have $f[x_0] = \Delta^0 f = f_0$ and $f[x_0, x_1] = (f_1 - f_0)/h$, so the assumption is correct for $k = 0, 1$. We now assume that the hypothesis is correct for some arbitrary k, and are to show that it then also is true for $k+1$. We have:

$$\begin{aligned} f[x_0, \ldots, x_k, x_{k+1}] &= \frac{f[x_1, \ldots, x_{k+1}] - f[x_0, \ldots, x_k]}{x_{k+1} - x_0} \\ &= \frac{\frac{1}{k!h^k}(\Delta^k f_1 - \Delta^k f_0)}{(k+1)h} = \frac{1}{(k+1)!h^{k+1}}\Delta^{k+1} f_0. \end{aligned}$$

b) Since $x = x_0 + sh$, we have $x_i = x_0 + ih$ so that

$$\prod_{i=1}^{k-1}(x - x_i) = h^k \prod_{i=0}^{k-1}(s - i) = h^k k! \binom{s}{k}.$$

c) Insert the results from **a)** and **b)** into the known formula

$$p_n(x) = \sum_{k=0}^{n} f[x_0, \ldots, x_k] \prod_{i=0}^{k-1}(x - x_i).$$

d) Sort the nodes in increasing order. Then we get the following table of forward differences:

$$\begin{array}{cccc} 1 & & & \\ & -1 & & \\ 0 & & 3 & \\ & 2 & & -3 \\ 2 & & 0 & \\ & 2 & & \\ 4 & & & \end{array}$$

and the polynomial becomes

$$p(x) = p(x_0+sh) = 1 - \binom{s}{1} + 3\binom{s}{2} - 3\binom{s}{3} = 1 - s + 3\frac{s(s-1)}{2} - 3\frac{s(s-1)(s-2)}{3!},$$

which is the same polynomial as in Task **1b)**, since in this case $x_0 = 0$ and $h = 1$.

Comment: Equivalently, it is possible to show *Newton's backward difference formula.* Backward differences on the sequence $\{f_n\}_{n=0}^{\infty}$ are defined by

$$\nabla^0 f_n = f_n, \qquad \nabla f_n = f_n - f_{n-1}, \qquad \nabla^k f_n = \nabla^{k-1} f_n - \nabla^{k-1} f_{n-1}, \qquad k = 1, 2, \ldots$$

Newton's backward difference formula is given by

$$p_n(x) = p_n(x_n + sh) = f_n + \sum_{k=1}^{n} (-1)^k \binom{-s}{k} \nabla^n f_n.$$

Sol:5 We have the quadrature rule on $[-1, 1]$

$$\int_{-1}^{1} f(x)\,dx \approx w_0 f(-\alpha) + w_1 f(\alpha),$$

with $0 < \alpha \leq 1$ The linearity of integration and the quadrature rule implies that the rule will be exact for all polynomials of degree 1 iff it is exact for a basis for this space. We chose the basis functions 1 and x. This results in the following two equations

$$\int_{-1}^{1} 1\,dx = 2 = w_0 + w_1,$$
$$\int_{-1}^{1} x\,dx = 0 = w_0(-\alpha) + w_1(\alpha) = \alpha(w_1 - w_0).$$

For any $0 < \alpha \leq 1$ the second equation gives $w_0 = w_1$, and then from the first $2w_0 = 2$. Thus the solution is $w_0 = w_1 = 1$, which is what we wanted to show.

If we want the formula to be exact for all polynomials of degree 2, we must include an additional basis function (any quadratic polynomial), so we get a basis for this space. Choosing x^2 we get the additional equation

$$\int_{-1}^{1} x^2\,dx = \frac{2}{3} = (-\alpha)^2 + \alpha^2 = 2\alpha^2$$

which has the unique positive solution $\alpha = 1/\sqrt{3}$.

To check if the formula is exact for any polynomial of degree 3, we simply check if it integrates exactly x^3, since we have then checked a basis for all cubic polynomials.

$$\int_{-1}^{1} x^3\,dx = 0,$$
$$\left(-\frac{1}{\sqrt{3}}\right)^3 + \left(\frac{1}{\sqrt{3}}\right)^3 = 0,$$

so the formula happens to be exact for all polynomials of degree 3 as well. Note that x^3 is an odd function, and it is easily seen that the quadrature formula is also exact for any odd function.

Sol:6 We write $E_1(f)$ and $E_2(f)$ for the error (exact integral minus approximation) for the trapezium rule and Simpson's rule on $\int_0^1 f\,dx$. It is trivial to compute

$$\int_{0}^{1} x^4\,dx = \frac{1}{5}, \qquad \int_{0}^{1} x^5\,dx = \frac{1}{6}.$$

The trapezium rule approximation for these integrals are

$$\int_{0}^{1} x^4\,dx \approx \frac{1}{2}\cdot 0^4 + \frac{1}{2}\cdot 1^4 = \frac{1}{2}, \qquad \int_{0}^{1} x^5\,dx \approx \frac{1}{2}\cdot 0^5 + \frac{1}{2}\cdot 1^5 = \frac{1}{2}.$$

So

$$E_1(x^4) = \frac{1}{5} - \frac{1}{2} = -\frac{3}{10}, \quad E_1(x^5) = \frac{1}{6} - \frac{1}{2} = -\frac{1}{3}.$$

Similarly for Simpson's rule the approximations are

$$\int_0^1 x^4\,dx \approx \frac{1}{6}\left(0^4 + 4\left(\frac{1}{2}\right)^4 + 1^4\right) = \frac{5}{24},$$

$$\int_0^1 x^5\,dx \approx \frac{1}{6}\left(0^5 + 4\left(\frac{1}{2}\right)^5 + 1^5\right) = \frac{3}{16}.$$

So

$$E_2(x^4) = \frac{1}{5} - \frac{5}{24} = -\frac{1}{120}, \quad E_2(x^5) = \frac{1}{6} - \frac{3}{16} = -\frac{1}{48}.$$

Demanding $E_1(x^5 - Cx^4) = 0$ we get

$$E_1(x^5 - Cx^4) = E_1(x^5) - CE_1(x^4) = -\frac{1}{3} + C\frac{3}{10} = 0,$$

which is trivially solved for $C = 10/9$.

The error requirement $|E_1(x^5 - Cx^4)| < |E_2(x^5 - Cx^4)|$ yields the inequality

$$\left|-\frac{1}{3} + C\frac{3}{10}\right| < \left|-\frac{1}{48} + C\frac{1}{120}\right|. \tag{1}$$

This inequality is obviously untrue for $C \leq 0$. Now $E_1(x^5 - Cx^4)$ is positive for $C > 10/9$ and $E_2(x^5 - Cx^4)$ is positive for $C > 5/2 > 10/9$. Thus for the region $0 < C \leq 10/9$, (1) is equivalent to

$$\begin{aligned} \frac{1}{3} - C\frac{3}{10} &< \frac{1}{48} - C\frac{1}{120}, \\ \frac{15}{48} &< C\frac{7}{24}, \\ C &> \frac{15}{14}, \end{aligned}$$

and is thus satisfied here for $15/14 < C \leq 10/9$.

For the region $10/9 < C \leq 5/2$, (1) is equivalent to

$$\begin{aligned} -\frac{1}{3} + C\frac{3}{10} &< \frac{1}{48} - C\frac{1}{120}, \\ C\frac{37}{120} &< \frac{17}{48}, \\ C &< \frac{85}{74}, \end{aligned}$$

and is therefore here satisfied for $10/9 < C < 85/74$.

Finally for the region $C > 5/2$

$$\begin{aligned} -\frac{1}{3} + C\frac{3}{10} &< -\frac{1}{48} + C\frac{1}{120}, \\ C\frac{7}{24} &< \frac{15}{48}, \\ C &< \frac{15}{14}. \end{aligned}$$

and the inequality is not satisfied for any values of C here.

Thus (1) is satisfied for $15/14 < C < 85/74$, which shows the desired result.