Norwegian University of Science
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**Exercise set 3:
Suggested solutions**

Sol:1 Rewrite the iteration scheme on the form

$$Q\mathbf{x}^{(k+1)} = (Q - A)\mathbf{x}^{(k)} + b$$

with

$$Q = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & -4 \end{bmatrix}, \qquad (Q - A) = \begin{bmatrix} -1 & -1 & 1 \\ 2 & 1 & -1 \\ -1 & 1 & -1 \end{bmatrix}, \qquad \text{and} \qquad b = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}$$

and $\mathbf{x}^{(k)} = [x_k, y_k, z_k]^T$. Find $T = Q^{-1}(Q - A)$ og A, and show $\|T\|_\infty = 0.75$. The iteration scheme converges for all starting values. Furthermore, $\lim_{k\to\infty} \mathbf{x}^{(k)} = \mathbf{x}$, where $\mathbf{x}$ is the solution of $A\mathbf{x} = b$.
The exact solution in this case is $\boldsymbol{x} = [1/9, 1/9, -4/3]$. You can find this by iterating until convergence, or by solving the system using Gaussian elimination.

By the use of Theorem 0.1 from the notes on nonlinear equations, using $D = \mathbb{R}^3$ and $L = \|T\|_\infty$, we get

$$\|\mathbf{x}^{(k)} - \mathbf{x}\|_\infty \leq \frac{\|T\|_\infty}{1 - \|T\|_\infty} \|\mathbf{x}^{(k)} - \mathbf{x}^{(k-1)}\|_\infty$$

or

$$\|\mathbf{x}^{(k)} - \mathbf{x}\|_\infty \leq \frac{\|T\|_\infty^k}{1 - \|T\|_\infty} \|\boldsymbol{x}^{(1)} - \mathbf{x}^{(0)}\|_\infty \leq 10^{-4}$$

Do one iteration to get $\mathbf{x}^{(1)}$, insert $\|T\|_\infty$, and see that $k = 37$ is sufficient. Such bounds are almost always very conservative, so in practice less iterations are needed.

Sol:2 **a)**

$$x^{(1)} = \begin{bmatrix} 1.6 \\ 0.5 \\ 1.26 \end{bmatrix}, \qquad x^{(2)} = \begin{bmatrix} 1.08 \\ 1.06 \\ 1.06 \end{bmatrix}, \qquad x^{(3)} = \begin{bmatrix} 0.96 \\ 1.03333 \\ 0.98267 \end{bmatrix}$$

The iterations seem to converge, which is reasonable since the matrix is strictly diagonally dominant. Furthermore, one can verify that the spectral radius of the iteration matrix is less than one, $\rho(T) < 1$.

b)

$$x^{(1)} = \begin{bmatrix} 1.6 \\ -5.3 \\ -17.3 \end{bmatrix}, \qquad x^{(2)} = \begin{bmatrix} 9.20 \\ -115.1 \\ -339.1 \end{bmatrix}, \qquad x^{(3)} = \begin{bmatrix} 153.07 \\ -2155.7 \\ -6317.0 \end{bmatrix}$$

The iterations diverge. This is because the spectral radius is greater than 1, so divergence is reasonable. The equations are the same, they are only permuted.

Sol:3 See exam paper: December 2008, problem 5.

Sol:4 Use your PYTHON codes as solutions.