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**Exercise set 2: Suggested solutions**

**Sol.1** It is sufficient to show the two conditions

$$G(D) \subseteq D \tag{1}$$

$$\max_i \sum_{j=1}^{3} \bar{g}_{ij} < 1, \qquad \text{where} \qquad \left| \frac{\partial g_i}{\partial x_j}(x) \right| \leq \bar{g}_{ij} \quad \text{for } x \in D. \tag{2}$$

It is relatively easy to see that

$$\begin{aligned} g_1(1,1,x_3) \approx 0.34 &< g_1(x_1,x_2,x_3) \leq 0.5 = g_1(0,x_2,x_3) \\ g_2(0,x_2,-1) \approx -0.048 &< g_2(x_1,x_2,x_3) < 0.09 \approx g_2(1,x_2,1) \\ g_3(-1,1,x_3) \approx -0.61 &< g_3(x_1,x_2,x_3) < -0.49 \approx g_2(1,1,x_3) \end{aligned}$$

so (1) is satisfied. Likewise, we can show that

$$\begin{array}{lll} \left|\dfrac{\partial g_1}{\partial x_1}\right| < 0.281 & \left|\dfrac{\partial g_1}{\partial x_2}\right| < 0.281 & \left|\dfrac{\partial g_1}{\partial x_3}\right| = 0 \\ \left|\dfrac{\partial g_2}{\partial x_1}\right| < 0.067 & \left|\dfrac{\partial g_2}{\partial x_2}\right| = 0 & \left|\dfrac{\partial g_2}{\partial x_3}\right| < 0.119 \\ \left|\dfrac{\partial g_3}{\partial x_1}\right| < 0.136 & \left|\dfrac{\partial g_3}{\partial x_2}\right| < 0.136 & \left|\dfrac{\partial g_3}{\partial x_3}\right| = 0 \end{array}$$

for all $x \in D$. This means that

$$\max_i \sum_{j=1}^{3} \bar{g}_{ij} = \max\{0.562, 0.186, 0.272\} = 0.562 < 1$$

so condition (2) is also satisfied. Test this numerically yourself.

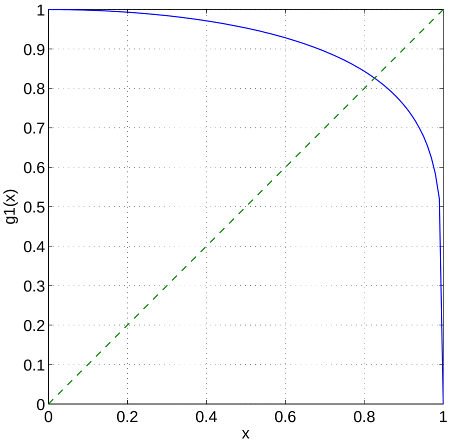
**Sol.2** The fixed point iterations are given by

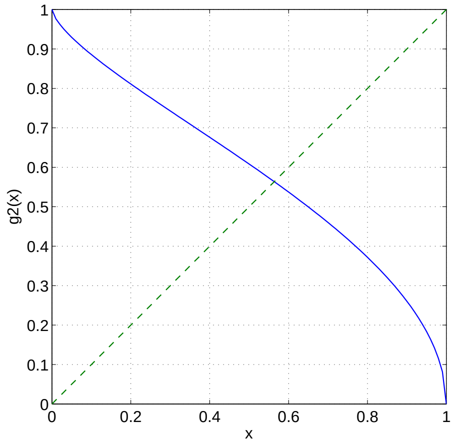
$$\begin{aligned} x_1^{(k+1)} &= \sqrt[3]{x_2^{(k)}} \qquad & x_1^{(k+2)} &= \sqrt[6]{1 - [x_1^{(k)}]^2} \\ x_2^{(k+1)} &= \sqrt{1 - [x_1^{(k)}]^2} \qquad & x_2^{(k+2)} &= \sqrt{1 - [x_2^{(k)}]^{2/3}} \end{aligned}$$

so we can view this as fixed point iterations on two scalar equations:

$$x = g_1(x) = \sqrt[6]{1 - x^2}, \qquad x = g_2(x) = \sqrt{1 - x^{2/3}}.$$

Start by locating the fixed points. This is easily done graphically:




This shows that $g_1$ has a fixed point near 0.8, and $g_2$ one near 0.5. For each of these, we must now find an interval $[a, b]$ so that *i)* $g_i([a, b]) \subseteq [a, b]$ and *ii)* $|g_i'(x)| < 1$ for $x \in [a, b]$.

Let us look at $g_1$ first. We see that

$$g_1'(x) = -\frac{x}{3(1-x^2)^{5/6}}, \qquad |g'(x)| < 1 \text{ for } 0 \leq x \leq 0.87.$$

But this interval does not satisfy *i)*. However, $g_1$ is monotonically decreasing on $[0, 0.87]$. After a little trial and error, we find

$$g_1([0.76, 0.87]) \subseteq [0.76, 0.87].$$

Similarly, we can show that the two conditions are satisfied for $g_2$ on the interval $[0.22, 0.80]$. Thus, we have proven that the equation has a unique fixed point in

$$D = \{x \in \mathbb{R}^2 : 0.76 \leq x_1 \leq 0.87, 0.22 \leq x_2 \leq 0.80\}$$

and the iterations converge for all starting values in this region.