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Exercise set 1: Solution hints

Sol:1 **a)** We would like to show that the error satisfies

$$\lim_{k\to\infty} \frac{|e_{k+1}|}{|e_k|^q} = C.$$

i) The root $x^\star = \arccos 0.5 \approx 1.0471975512$, and

k	x_k	$\lvert e_k\rvert$	$\lvert e_{k+1}\rvert / \lvert e_k\rvert$	$\lvert e_{k+1}\rvert / \lvert e_k\rvert^2$
0	0.5000000000	$5.47 \cdot 10^{-1}$	$4.39 \cdot 10^{-1}$	0.803
1	1.2875729002	$2.40 \cdot 10^{-1}$	$4.44 \cdot 10^{-2}$	0.185
2	1.0578736992	$1.07 \cdot 10^{-2}$	$3.03 \cdot 10^{-3}$	0.283
3	1.0472298506	$3.23 \cdot 10^{-5}$	$9.32 \cdot 10^{-6}$	0.287
4	1.0471975514	$3.01 \cdot 10^{-10}$	$8.69 \cdot 10^{-11}$	0.287
5	1.0471975512	$2.62 \cdot 10^{-20}$	$7.56 \cdot 10^{-21}$	0.287
6	1.0471975512	$1.98 \cdot 10^{-40}$		

As expected, we have quadratic convergence, i.e. $q = 2$, with $C = 0.287$ (in this case, the calculations have been done in Maple with accuracy of over 50 digits).

ii) The root $x^\star = 0$, and

k	x_k	$\lvert e_k\rvert$	$\lvert e_{k+1}\rvert / \lvert e_k\rvert$	$\lvert e_{k+1}\rvert / \lvert e_k\rvert^2$
1	0.5000000000	$5.00 \cdot 10^{-1}$	0.54	3.69
2	0.2707470413	$2.71 \cdot 10^{-1}$	0.52	7.07
3	0.1414747338	$1.41 \cdot 10^{-1}$	0.51	13.81
4	0.0724047358	$7.24 \cdot 10^{-2}$	0.51	27.29
5	0.0366392002	$3.66 \cdot 10^{-2}$	0.50	54.26
6	0.0184314669	$1.84 \cdot 10^{-2}$	0.50	108.18
7	0.0092440432	$9.24 \cdot 10^{-3}$	0.50	216.02
8	0.0046291426	$4.63 \cdot 10^{-3}$	0.50	431.71
9	0.0023163571	$2.32 \cdot 10^{-3}$	0.50	863.09
10	0.0011586257	$1.16 \cdot 10^{-3}$		

In this case the convergence is linear, with constant $C = 0.5$. This is caused by $f'(0)$ being zero, so the condition for quadratic convergence is not satisfied.

iii) The root $x^\star = 0$, and

k	x_k	$\lvert e_k \rvert$	$\lvert e_{k+1}\rvert/\lvert e_k\rvert$	$\lvert e_{k+1}\rvert/\lvert e_k\rvert^2$
1	0.5000000000	$5.00 \cdot 10^{-1}$	0.66	3.02
2	0.3309759368	$3.31 \cdot 10^{-1}$	0.66	4.55
3	0.2199738473	$2.20 \cdot 10^{-1}$	0.67	6.83
4	0.1464514253	$1.46 \cdot 10^{-1}$	0.67	10.25
5	0.0975760249	$9.76 \cdot 10^{-2}$	0.67	15.38
6	0.0650334672	$6.50 \cdot 10^{-2}$	0.67	23.07
7	0.0433505497	$4.34 \cdot 10^{-2}$	0.67	34.60
8	0.0288988576	$2.89 \cdot 10^{-2}$	0.67	51.91
9	0.0192654581	$1.93 \cdot 10^{-2}$	0.67	77.86
10	0.0128435063	$1.28 \cdot 10^{-2}$		

This time the convergence is linear with $C = 0.67$. The reason is the same as in *ii)*.

b) *i)* $x^\star = \arccos(0.5) = \pi/3$, $f'(x^\star) = -\sqrt{3}/2$, so this root has multiplicity 1.

ii) $x^\star = 0$, and $f'(0) = 0$, $f''(0) = 1$. The root has multiplicity 2.

iii) $x^\star = 0$, and $f'(0) = f''(0) = 0$, $f'''(0) = 3$. This root has multiplicity 3.

c) From the definition of multiplicity in the text, we can write

$$\mu(x) = \frac{(x - x^\star)^m q(x)}{m(x - x^\star)^{m-1} q(x) + (x - x^\star)^m q'(x)} = (x - x^\star)\frac{q(x)}{mq(x) - (x - x^\star)q'(x)}.$$

So $x^\star$ is a simple root of $\mu(x)$ since $q(x^\star) \neq 0$. We find Newton's method applied to $\mu(x)$ as

$$x_{k+1} = x_k - \frac{\mu(x_k)}{\mu'(x_k)} = x_k - \frac{f(x_k) f'(x_k)}{[f'(x_k)]^2 - f(x_k) f''(x_k)}$$

which converges quadratically.

d) You may do this task yourself. Notice that rounding errors can be a problem here, since $f(x)$ and $f'(x)$ both tend to zero when x_k tends to $x^\star$.

e) This task is similar enough to Newton's method that you should be able to do it on your own.

Sol:2 **a)** We rewrite the system of equations as

$$F(X) = \begin{bmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{bmatrix} = \begin{bmatrix} x_1^2 + x_2^2 - 1 \\ x_1^3 - x_2 \end{bmatrix} = 0,$$

where $X = (x_1, x_2)^T$. The Jacobian matrix becomes

$$J(X) = \begin{bmatrix} \partial f_1/\partial x_1 & \partial f_1/\partial x_2 \\ \partial f_2/\partial x_1 & \partial f_2/\partial x_2 \end{bmatrix} = \begin{bmatrix} 2x_1 & 2x_2 \\ 3x_1^2 & -1 \end{bmatrix}.$$

We can then write Newton's method as

$$X^{(n+1)} = X^{(n)} + H^{(n)},$$

where $H^{(n)}$ is implicitly given by

$$J(X^{(n)})H^{(n)} = -F(X^{(n)}). \tag{1}$$

We must avoid initial values where the Jacobian is singular, $(\det\big(J(X)\big) = 0)$:

$$\det\big(J(X)\big) = -2x_1 - 6x_1^2x_2 = -2x_1(1 + 3x_1x_2) = 0.$$

Thus, we must keep away from the curves $x_1 = 0$ and $3x_1x_2 = -1$, and choose e.g. initial values $x_1^{(0)} = x_2^{(0)} = 0.5$. Thus

$$F(X^{(0)}) = \begin{bmatrix} -0.5 \\ -0.375 \end{bmatrix}$$

$$J(X^{(0)}) = \begin{bmatrix} 1 & 1 \\ 0.75 & -1 \end{bmatrix}$$

and to obtain $H^{(0)}$ we solve the linear system of equations

$$\begin{bmatrix} 1 & 1 \\ 0.75 & -1 \end{bmatrix} \begin{bmatrix} h_1^{(0)} \\ h_2^{(0)} \end{bmatrix} = \begin{bmatrix} 0.5 \\ 0.375 \end{bmatrix}.$$

The solution is

$$H^{(0)} = \begin{bmatrix} 0.5 \\ 0 \end{bmatrix}.$$

After one iteration we thus get $x_1^{(1)} = 1$ and $x_2^{(1)} = 0.5$.
In the same way, we obtain $x_1^{(2)} = 0.85$ and $x_2^{(1)} = 0.55$.
In our case we can alternatively easily calculate J^{-1} by hand, which leads to the explicit iteration

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \mapsto \frac{1}{2x_1(1 + 3x_1x_2)} \begin{bmatrix} x_1^2 + 4x_1^3x_2 + x_2^2 + 1 \\ x_1^2(3x_2^2 - x_1^2 + 3) \end{bmatrix}.$$

Calculating J^{-1} as we have done will normally be very cumbersome. Instead one usually solves (1) numerically, e.g. with the conjugate gradient method. Python does this for us if we solve (1) using the \ operator.

b) See the Python programs on the homepage.

c) As we saw in **a)**, the Jacobian is singular on the x_1 axis. This causes the algorithm to fail, since we don't get a unique solution when solving (1).

Sol:3 We consider the sequence specified in Definition 1.4 with $\varepsilon_k = |x_k - 0| = x_k$.

$$\begin{aligned} \frac{x_{k+1}}{x_k} &= \frac{2^{-(k+1)^\alpha}}{2^{-k^\alpha}} \\ &= 2^{k^\alpha - (k+1)^\alpha} \end{aligned}$$

We consider the exponent and rewrite it by the generalized binomial series

$$\begin{aligned} k^\alpha - (k+1)^\alpha &= k^\alpha - k^\alpha \left(1 + \frac{1}{k}\right)^\alpha \\ &= k^\alpha - k^\alpha \left(1 + \alpha k^{-1} + \binom{\alpha}{2} k^{-2} + \cdots\right) \\ &= -\alpha k^{\alpha-1} - \binom{\alpha}{2} k^{\alpha-2} - \cdots \end{aligned}$$

The series converges for $k > 1$, and we see that the dominating term when $k \to \infty$ is $-\alpha k^{\alpha-1}$. Therefore

$$\lim_{k\to\infty} k^\alpha - (k+1)^\alpha = \begin{cases} 0, & \alpha < 1, \\ -1, & \alpha = 1, \\ -\infty & \alpha > 1. \end{cases}$$

Since 2^x is continuous for all $x \in \mathbb{R}$ and $\lim_{x\to-\infty} 2^x = 0$,

$$\mu = \lim_{k\to\infty} \frac{x_{k+1}}{x_k} = \begin{cases} 1, & \alpha < 1, \\ \frac{1}{2}, & \alpha = 1, \\ 0, & \alpha > 1. \end{cases}$$

According to the definition, the sequence (x_k) converges sublinearly when $\alpha < 1$, linearly when $\alpha = 1$ and superlinearly when $\alpha > 1$