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TMA4215 Numerical Mathematics Autumn 2017

Exercise set 7

1 Given an ordinary differential equation

$$y' = f(t, y), \qquad y(t_0) = y_0, \qquad t_0 \le t \le t_{\text{end}}. \tag{1}$$

You can assume that f satisfies the Lipschitz condition

$$\|f(t, y) - f(t, \tilde{y})\| \le L\|y - \tilde{y}\|.$$

A *one-step method* for solving this differential equation can be described by

$$y_{n+1} = y_n + h\Phi(t_n, y_n; h), \qquad n = 0, 1, \ldots, N-1, \quad h = \frac{t_{\text{end}} - t_0}{N} \tag{2}$$

Assume the following:

- The local truncation error given by

  $$d_{n+1} = y(t_{n+1}) - y(t_n) - h\Phi\big(t_n, y(t_n); h\big)$$

  satisfies

  $$\|d_{n+1}\| \le Dh^{p+1}$$

  where D is a positive constant.
- The function Φ is Lipschitz continuous, with Lipschitz constant M, i.e.

  $$\|\Phi(t_n, y; h) - \Phi(t_n, \tilde{y}; h)\| \le M\|y - \tilde{y}\|. \tag{3}$$

a) Show that in this case, the global error in t_{end} satisfies

$$\|e_N\| = \|y(t_{\text{end}}) - y_N\| \le Ch^p,$$

where C is a positive constant depending on M, D and the interval $t_{\text{end}} - t_0$.

b) Assume that a two-stage explicit Runge–Kutta method given by the Butcher tableau

$$\begin{array}{c|cc} 0 & & \\ c_2 & c_2 & \\ \hline & b_1 & b_2 \end{array}$$

is used to solve (1). Show that the method can be written on the form (2). Now assume that $h \le h_{\max}$ and show that Φ satisfies the Lipschitz condition in y, with Lipschitz constant M that depends on the method coefficients c_2, b_1 and b_2, as well as L and $h_{\max}$.

2 Kutta's method from 1901 is the most famous of all explicit Runge–Kutta pairs, given by the following Butcher tableau:

$$\begin{array}{c|cccc} 0 & & & & \\ \frac{1}{2} & \frac{1}{2} & & & \\ \frac{1}{2} & 0 & \frac{1}{2} & & \\ 1 & 0 & 0 & 1 & \\ \hline & \frac{1}{6} & \frac{1}{3} & \frac{1}{3} & \frac{1}{6} \end{array}$$

a) Verify that the method has order 4 by checking all 8 order conditions.

b) An alluring thought is to now find a new set of weights, say $\hat{b}_s$ such that the accompanying method is of order 3, for error estimates and step length control. Try to find such a set of $\hat{b}_s$.

3 **a)** Find the eigenvalues of the matrix

$$M = \begin{pmatrix} -10 & -10 \\ 40 & -10 \end{pmatrix}.$$

b) Assume that you are to solve the differential equation

$$y' = My, \qquad y(0) = y_0$$

using the improved Euler method. What is the largest step size $h_{\max}$ you can use?

c) Solve the equation

$$y' = My + g(t), \qquad 0 \leq t \leq 10$$

with

$$g(t) = \big(\sin(t), \cos(t)\big)^T, \qquad y(0) = \left(\frac{5210}{249401}, \frac{20259}{249401}\right)^T$$

by using the improved Euler method. Choose step sizes a little smaller than and a little larger than $h_{\max}$. What do you observe?