TMA4215
Numerical Mathematics
Autumn 2017

Norwegian University of Science and Technology
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Exercise set 6

1 We will study *Hermite interpolation* in this task.

Given $n+1$ distinct nodes $x_0, x_1, \dots, x_n$, we are to find a polynomial $p(x)$ of lowest possible degree satisfying

$$p(x_i) = y_i, \quad p'(x_i) = v_i, \quad i = 0, 1, \dots, n \tag{1}$$

for arbitrary values y_i and v_i.

a) Why is it reasonable to assume that $p(x)$ will be of degree less than or equal to $2n+1$, i.e. $p \in \mathbb{P}_{2n+1}$?

b) Show that a function $g(x)$ given by

$$g(x) = \sum_{i=0}^{n} y_i A_i(x) + \sum_{i=0}^{n} v_i B_i(x)$$

satisfies the conditions (1) if the functions $A_i(x)$ and $B_i(x)$ satisfy

$$\begin{aligned} A_i(x_j) &= \delta_{ij}, \quad & B_i(x_j) &= 0, \\ A_i'(x_j) &= 0, \quad & B_i'(x_j) &= \delta_{ij} \end{aligned} \tag{2}$$

where $\delta_{ij} = 1$ when $j = i$ and else is 0.

c) Let $L_i(x)$ be the ordinary cardinal functions in Lagrange interpolation. Show that the following polynomials satisfy (2) for all $i = 0, 1, \dots, n$:

$$A_i(x) = \big(1 - 2(x - x_i)L_i'(x_i)\big)L_i^2(x), \quad B_i(x) = (x - x_i)L_i^2(x).$$

d) Use this to find a third degree polynomial $p(x)$ satisfying

$$\begin{aligned} p(1) &= 1, \quad & p(2) &= 14 \\ p'(1) &= 3, \quad & p'(2) &= 24. \end{aligned}$$

2 Construct an adaptive trapezoid algorithm. Apply the algorithm to find the value of the *Fresnel integral*

$$S(x) = \int_0^x \sin(t^2)\,\mathrm{d}t.$$

for $x = 0.8$. Use $tol = 2 \cdot 10^{-3}$.

3 **a)** Find an approximation to the integral

$$\int_{-1}^{1} \frac{e^x}{\sqrt{1-x^2}}\,dx$$

by using Gaussian quadrature with $n = 1$ (two nodes). Use the Gaussian quadrature based on the Legendre polynomials.

b) Find a Gaussian quadrature of the form

$$\int_{-1}^{1} \frac{f(x)}{\sqrt{1-x^2}}\,dx \approx A_1 f(x_1) + A_2 f(x_2)$$

and use this to calculate the integral in **a)**. Compare with the exact solution.

c) Find an upper limit for the error in **b)**.

4 Find the first three Laguerre polynomials, i.e. polynomials that are orthogonal with respect to the inner product

$$\langle p, q\rangle = \int_0^\infty e^{-x} p(x) q(x)\,dx.$$

5 Show that the polynomials defined by

$$\Phi_k(x) = \frac{1}{2^k\,k!}\frac{d^k}{dx^k}\left[(x^2-1)^k\right]$$

are orthogonal with respect to the inner product

$$\langle p, q\rangle = \int_{-1}^{1} p(x) q(x)\,dx.$$

Hint: Note that the j-th derivative of $(x^2-1)^k$ is divisible by (x^2-1) if $j < k$. Use partial integration repeatedly.

6 Exam December 2008, Problem 3

Note: S&M,
Süli, Endre, and David F. Mayers. An introduction to numerical analysis. Cambridge university press, 2003.