Norwegian University of Science and Technology
Department of Mathematical Sciences

TMA4215 Numerical Mathematics
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Exercise set 5

1 **a)** Use divided differences and Newton's interpolation formula to find the interpolating polynomial of lowest possible degree for the points in the table:

x_i	0	2	3
y_i	1	2	4

b) Add the point $(1, 0)$ to the table in **a)**. What is the new interpolating polynomial?

2 In this task, you are to approximate $\sin(x)$ on the interval $[0, \pi]$ using polynomial interpolation.

a) Choose 4 equidistant nodes on the interval. Find the polynomial and an upper bound for the error $|\sin(x) - p_3(x)|$.

b) Repeat **a)**, but use Chebyshev nodes instead (see hint). (You do not have to calculate the polynomial, but show the interpolation points.)

c) In the two cases (equidistant and Chebyshev nodes), find an expression for an upper error bound, expressed by n (the polynomial degree). Plot the error bound as a function of n and compare the two cases.

Hint: The Chebyshev nodes are defined on the interval $[-1, 1]$. To move them over to another interval $[a, b]$, the change of variables

$$x = \frac{a+b}{2} + \frac{b-a}{2}t, \qquad t \in [-1, 1]$$

is used. The error formula is adjusted in the same way.

3 Write two PYTHON functions, one that calculates the table of divided differences based on a given dataset, and one that calculates the value of the interpolating polynomial in given points, based on this table.
Iterated multiplication may be useful in this task. Let

$$p(x) = f_0 + f_1(x - x_0) + f_2(x - x_0)(x - x_1) + \cdots + f_n(x - x_0)(x - x_1)\cdots(x - x_{n-1}),$$

and define the polynomials $b_n, b_{n-1}, \ldots, b_0$ by

$$\begin{aligned} b_n(x) &= f_n \\ b_{n-1}(x) &= f_{n-1} + (x - x_{n-1})b_n(x) \\ &\vdots \\ b_0(x) &= f_0 + (x - x_0)b_1(x) \end{aligned}$$

Then $b_0(x) = p(x)$. This equality is easily seen by substituting for $b_1(x), b_2(x), \ldots, b_n(x)$ in the expression for b_0 and expand the expression. Test the functions on the dataset in Task 1.

4 Let the distance h between nodes be such that $x_i = a + ih$, $i = 0, 1, 2, \ldots$ Let $f_i = f(x_i)$, $i = 0, 1, 2, \ldots$
On such a sequence $\{f_i\}_{i=0}^{\infty}$, we can define a *forward difference* recursively by

$$\Delta^0 f_0 = f_0, \quad \Delta f_0 = f_1 - f_0, \qquad \Delta^k f_0 = \Delta(\Delta^{k-1} f_0) = \Delta^{k-1} f_1 - \Delta^{k-1} f_0, \qquad k = 1, 2, \ldots$$

We get

$$\Delta^2 f_0 = f_2 - 2f_1 + f_0, \qquad \Delta^3 f_0 = f_3 - 3f_2 + 3f_1 - f_0, \qquad \text{and so on.}$$

Let $x = x_0 + sh$, where $s \in \mathbb{R}$. The task is about showing that the polynomial interpolating f in the nodes x_i, $i = 0, 1, \ldots, n$ can be written

$$p_n(x) = p_n(x_0 + sh) = f_0 + \sum_{k=1}^{n} \binom{s}{k} \Delta^k f_0 \tag{1}$$

where

$$\binom{s}{k} = \frac{s(s-1)\cdots(s-k+1)}{k!}.$$

a) Show by induction:

$$f[x_0, x_1, \ldots, x_k] = \frac{1}{k!h^k} \Delta^k f_0.$$

b) Show that

$$\prod_{i=0}^{k-1} (x - x_i) = k!h^k \binom{s}{k}, \qquad k \geq 1.$$

c) Use the results from **a)** and **b)** to prove Newton's forward difference formula (1).

d) Apply the formula to the dataset in Task **1b)**.

Comment: Equivalently, it is possible to show *Newton's backward difference formula.* Backward differences on the sequence $\{f_n\}_{n=0}^{\infty}$ are defined by

$$\nabla^0 f_n = f_n, \qquad \nabla f_n = f_n - f_{n-1}, \qquad \nabla^k f_n = \nabla^{k-1} f_n - \nabla^{k-1} f_{n-1}, \qquad k = 1, 2, \dots$$

Newton's backward difference formula is given by

$$p_n(x) = p_n(x_n + sh) = f_n + \sum_{k=1}^{n} (-1)^k \binom{-s}{k} \nabla^n f_n.$$

5 Exercise 7.3, chapter 7 from S&M book

6 Exercise 7.6, chapter 7 from S&M book

Note: S&M,
Süli, Endre, and David F. Mayers. An introduction to numerical analysis. Cambridge university press, 2003.