1 Given the iteration scheme:

$$\begin{aligned} 4x_{k+1} &= -x_k - y_k + z_k + 2 \\ 6y_{k+1} &= 2x_k + y_k - z_k - 1 \\ -4z_{k+1} &= -x_k + y_k - z_k + 4 \end{aligned}$$

Prove that $\mathbf{x}^{(k)} = [x_k, y_k, z_k]^T$ converges to a limit $\mathbf{x}$ for all starting values $\mathbf{x}^{(0)}$ when $k \to \infty$. What is the limit $\mathbf{x}$?

How many iterations are needed to ensure $\|\mathbf{x}^{(k)} - \mathbf{x}\|_\infty \leq 10^{-4}$ if $\mathbf{x}^{(0)} = [0,0,0]^T$? Is this number realistic, or do you think that less iterations are needed in practice?

2 Solve the two systems of equations by Gauss–Seidel iterations:

a)

$$\begin{aligned} 3x + y + z &= 5 \\ x + 3y - z &= 3 \\ 3x + y - 5z &= -1 \end{aligned}$$

b)

$$\begin{aligned} 3x + y + z &= 5 \\ 3x + y - 5z &= -1 \\ x + 3y - z &= 3. \end{aligned}$$

Use $[0.1, 0.1, 0.1]^T$ as the starting point. Comment on the results. Do they comply with theory?

3 Exam problem: December 2008, problem 5.

4 Given the linear equation

$$\begin{aligned} 4x_1 - x_2 - x_4 &= 0 \\ -x_1 + 4x_2 - x_3 - x_5 &= 5 \\ -x_2 + 4x_3 - x_6 &= 0 \\ -x_1 + 4x_4 - x_5 &= 6 \\ -x_2 - x_4 + 4x_5 - x_6 &= -2 \\ -x_3 - x_5 + 4x_6 &= 6 \end{aligned}$$

This equation is to be solved using the successive overrelaxation (SOR) method. This is a simple extension of the Gauss-Seidel method, where the iterations use a weighted component average of the previous iteration value, and the Gauss-Seidel iterate. Suppose we are looking at an $n \times n$ system $\mathbf{Ax} = \mathbf{b}$. The SOR method, for a given value of the relaxation parameter $\omega \in (0, 2)$, and initial value $\mathbf{x}^{(0)}$, then becomes in component form

$$x_i{}^{(k+1)} = \frac{\omega}{a_{ii}} \left(b_i - \sum_{j=1}^{i-1} a_{ij} x_j^{(k+1)} - \sum_{j=i+1}^{n} a_{ij} x_j^{(k)} \right) + (1 - \omega) x_i{}^{(k)}, \quad k = 0, 1, 2, \ldots$$

for $i = 1, 2, \ldots, n-1, n$.
Note that $\omega = 1$ gives the Gauss-Seidel method. $\omega > 1$ is used to accelerate convergence from Gauss-Seidel. Let T_ω define the iteration matrix for the method, so that we can write:

$$\mathbf{x}^{(k+1)} = T_\omega \mathbf{x}^{(k)} + \mathbf{c}_\omega, \quad k = 0, 1, 2, \ldots$$

The task amounts to

a) Finding the optimal value of the relaxation parameter ω, and accompanying $\rho(T_\omega)$. Write a function for calculating ρ, and plot $\rho(T_\omega)$ as a function of ω.

b) Do 10 iterations for this choice of ω, and for each iteration print the error

$$\|\mathbf{x}^{(k)} - \mathbf{x}\|_2.$$

c) Repeat **b)** for different choices of ω, for example 1.0 and 1.3. How does the convergence compare to the one in **b)**? Is the behaviour as we would expect? Determine also the value of ω for which $\rho(T_\omega) = 1$, and iterate with choices of ω around this value. Are the results consistent with the theory?